

On Friendly Index Set of Hypercube Q_n

Ebrahim Salehi

Department of Mathematical Sciences
University of Nevada, Las Vegas
Las Vegas, NV 89119

United States of America
ebrahim.salehi@unlv.edu

Donald McGinn

Department of Mathematics and Statistics
University of West Florida
Pensacola, FL 32514

United States of America
dmcginn@uwf.edu

Abstract

For a graph $G = (V, E)$, a binary vertex labeling (coloring) $f : V(G) \rightarrow \mathbb{Z}_2$, is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. The friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f_* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f_*(xy) = |f(x) - f(y)|$ for all $xy \in E(G)$. Let $e_f(i) = |f_*^{-1}(i)|$ be the number of edges labeled i . The friendly index set (or cordial set) of the graph G , denoted by $C(G)$, is defined by

$$C(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly vertex labeling of } G\}.$$

In this paper, among other facts, we investigate the friendly index set of n -cubes Q_n and improve the conjecture that is stated in [15]. In addition, without computer aid, we completely determine the friendly index sets of Q_n for $n \leq 6$.

Key Words: Hypercube; friendly labeling; Cordial graphs; friendly index set.

AMS Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [4]. Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}_2$ be a binary vertex labeling (or coloring) of G . For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The labeling f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. Any friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f_* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f_*(xy) = |f(x) - f(y)|$ for

all $xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f_*^{-1}(i)|$ be the number of edges of G that are labeled i . The number $N(f) = |e_f(1) - e_f(0)|$ is called the *cordial index* of f . A graph G is said to be *cordial* if it admits a friendly labeling with cordial index 0 or 1.

To illustrate the above concepts, consider the graph G of Figure 1, which has eight vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that the four vertices be labeled 0 and the other four 1.

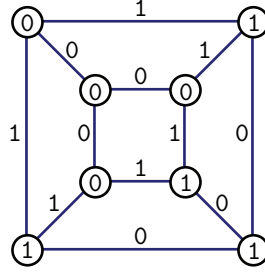


Figure 1: An Example of friendly labeling with cordial index 0.

Figure 1 also shows the associated edge labeling of G , where six edges have labels 1 while the other six edges have labels 0. Therefore, the cordial index of this labeling is $6 - 6 = 0$.

In 1978, I. Cahit [1] introduced the concept of cordial labeling as a weakened version of the less tractable graceful and harmonious labelings. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. Hovay [6] later generalized the concept of cordial graphs and introduced A -cordial labelings, where A is an abelian group. A graph G is said to be A -cordial if it admits a labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cairnie-Edwards [2] proved that the problem of deciding whether or not a graph G is cordial is NP-complete, as conjectured by Kirchherr [8]. They specifically proved that for a graph $G = (V, E)$ with even number of edges, the question of whether G admits a cordial labeling is NP-complete.

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that is mentioned in the bibliography section, including [5, 7, 9, 10, 11, 12, 13, 14].

In [3] Chartrand-Lee-Zhang introduced the concept of *friendly index set* of a graph G . The

friendly index set or *cordial set* of the graph G , denoted by $C(G)$, is defined by

$$C(G) = \{N(f) : f \text{ is a friendly labeling of } G\}.$$

Note that if 0 or 1 is in $C(G)$, then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. For the graph G in Figure 1, it is easy to verify that $C(G) = \{0, 4, 12\}$. The friendly colorings of G that provide the other cordial indices are presented in Figure 2.

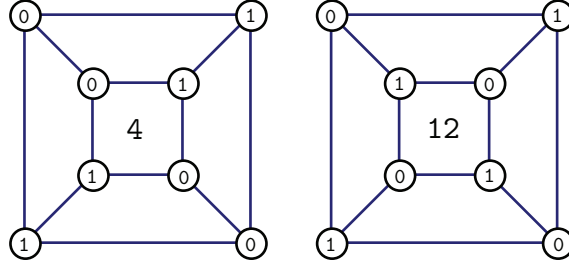


Figure 2: Two friendly labelings of G with indices 4 and 12.

In this paper, we focus on the group $A = \mathbb{Z}_2$ and, among other things, determine the friendly index sets of hypercubes Q_n for $n \leq 6$. A friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ is called a *maximum friendly labeling* of G if its friendly index is maximal, that is, $N(f) = |E(G)|$. In this case, we call $N(f)$ the maximum friendly index of G . Also, if $f : V(G) \rightarrow \mathbb{Z}_2$ is a friendly labeling, so is its inverse labeling $g : V(G) \rightarrow \mathbb{Z}_2$ defined by $g(v) = 1 - f(v)$ for all $v \in V(G)$. Moreover, $N(g) = N(f)$.

For a graph $G = (p, q)$ of order p , size q , and a friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of G , we have

$$N(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)| = |q - 2e_f(0)|. \quad (1.1)$$

Therefore, to find the cordial index of f it is enough to find $e_f(1)$ or $e_f(0)$. Moreover, to determine the cordial set of G it is enough to compute $e_f(1)$ [or $e_f(0)$] for different friendly labelings.

Another immediate consequence of Equation (1.1) is the following useful fact:

Observation 1.1. *If G is a graph G of size q , then $C(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.*

2 Hypercubes

Hypercubes (or n -cubes), denoted by Q_n , are defined by $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ ($n \geq 2$).

In this section, some of the properties of hypercubes Q_n that are utilized for finding its cordial set are investigated. Clearly, Q_n has 2^n vertices, $n2^{n-1}$ edges, and is an n -regular graph.

Remark 2.1. Let $\{u_i : 1 \leq i \leq 2^{n-1}\}$ be the vertices of Q_{n-1} and $V(K_2) = \{a, b\}$. Also, let H be the subgraph of Q_n induced by the vertices $\{v_i = (u_i, a) : 1 \leq i \leq 2^{n-1}\}$ and K be the subgraph induced by the vertices $\{w_i = (u_i, b) : 1 \leq i \leq 2^{n-1}\}$. These two subgraphs are isomorphic with Q_{n-1} , and Q_n is the disjoint union of H and K with the added edges $e_i = v_i w_i$ ($1 \leq i \leq 2^{n-1}$).

Remark 2.2. The cube Q_n also can be viewed as a graph whose vertices are labeled by the binary n -tuples $(a_1, a_2, \dots, a_n)$ [that is $a_i = 0$ or 1 for $1 \leq i \leq n$] such that two vertices are adjacent if and only if their corresponding n -tuples differ in exactly one position. Note that the hypercube Q_n is a bipartite graph. To every vertex $v = (a_1, a_2, \dots, a_n) \in Q_n$, we assign a number $n(v) = a_1 + a_2 + \dots + a_n$. Then the two partite sets are $S_n = \{v \in Q_n : n(v) \equiv 0 \pmod{2}\}$ and $T_n = \{v \in Q_n : n(v) \equiv 1 \pmod{2}\}$.

Observation 2.3. Let $a \in S_n$ and $b \in T_n$ be two vertices of Q_n with the binary representations $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, respectively. Then

- (1) $a \times 0 = (a_1, \dots, a_n, 0) \in S_{n+1}$ and $b \times 0 = (b_1, \dots, b_n, 0) \in T_{n+1}$.
- (2) If $a_n = b_n = 0$, then $(a_1, \dots, a_{n-1}, 1, 1) \in S_{n+1}$, $(b_1, \dots, b_{n-1}, 1, 1) \in T_{n+1}$.
- (3) If $a_n = b_n = 1$, then $(a_1, \dots, a_{n-1}, 0, 1) \in S_{n+1}$, $(b_1, \dots, b_{n-1}, 0, 1) \in T_{n+1}$.
- (4) If $a_n = 0$ & $b_n = 1$, then $(a_1, \dots, a_{n-1}, 1, 1) \in S_{n+1}$, $(b_1, \dots, b_{n-1}, 0, 1) \in T_{n+1}$.

Moreover, in each case the two new vertices of Q_{n+1} have the same adjacency relations as that of a and b in Q_n .

Definition 2.4. We define $A : V(Q_n) \times V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$A(u_i, v_j) = \begin{cases} 1 & \text{if } u_i v_j \in E(Q_n) \\ 0 & \text{if } u_i v_j \notin E(Q_n). \end{cases}$$

Theorem 2.5. For $n \geq 3$, there are 2^{n-3} pair of non-adjacent vertices u_i, v_i ($1 \leq i \leq 2^{n-3}$) in Q_n such that $u_i \in S_n$, $v_i \in T_n$ and $A(u_i, v_j) = 0$ ($1 \leq i, j \leq 2^{n-3}$).

Proof. We proceed by induction on n . For $n = 3$, these two vertices are $u_1 = (0, 0, 0)$, $v_1 = (1, 1, 1)$. Suppose the statement of theorem is true for Q_n . Then by Observation 2.3, $(u_i \times 0)$, $(v_i \times 0)$ would provide 2^{n-3} such pairs of vertices for Q_{n+1} . The other 2^{n-3} pairs of vertices for Q_{n+1} can be obtain from (u_i, v_i) by replacing their last coordinates with “1, 1” or “0, 1” depending on whether that last coordinates are 0 or 1, respectively. Table 1 illustrates this process from Q_3 to Q_4 then to Q_5 . $\square$

Remark 2.6. Clearly, the independent number of Q_n is 2^{n-1} , the cardinality of each partite sets S_n or T_n . However, Theorem 2.5 provides an independent set of vertices with cardinality 2^{n-2} half selected from each partite set. Furthermore, half of these independent vertices belong to $Q_{n-1} \times \{0\}$ and the other half belong to $Q_{n-1} \times \{1\}$.

Q_3		Q_4		Q_5	
u_i	v_i	u_i	v_i	u_i	v_i
(0,0,0)	(1,1,1)	(0,0,0,0)	(1,1,1,0)	(0,0,0,0,0)	(1,1,1,0,0)
				(0,0,0,1,1)	(1,1,1,1,1)
		(0,0,1,1)	(1,1,0,1)	(0,0,1,1,0)	(1,1,0,1,0)
				(0,0,1,0,1)	(1,1,0,0,1)

Table 1: The induction steps of Theorem 2.5, from Q_3 to Q_4 to Q_5 .

Remark 2.7. Theorem 2.5 guarantees the existence of 2^{n-2} independent vertices with half of them from S_n and the other half from T_n . However, there might be more such independent vertices. For example, Table 2 present 20 independent vertices with the property that 10 are from S_6 and another 10 are from T_6 .

$u \in S_6$	$v \in T_6$
(0,0,0,0,0,0)(0,0,0,0,1,1)(0,0,0,1,1,0)	(0,1,1,1,0,0)(0,1,0,1,0,1)(0,1,1,0,0,1)
(0,0,1,0,1,0)(0,1,0,0,1,0)(1,1,0,0,0,0)	(1,0,1,0,0,1)(1,0,1,1,0,0)(0,1,1,1,1,1)
(0,1,0,0,1,0)(1,0,0,0,1,0)(1,1,0,0,1,1)	(1,0,1,1,1,1)(1,0,0,1,0,1)(0,0,1,1,0,1)
(1,1,1,0,1,0)	(1,1,1,1,0,1)

Table 2: 20 independent vertices of Q_6 , with 10 from S_6 and 10 from T_6 .

Theorem 2.8. For $n \geq 3$, there are 2^{n-2} pairwise disjoint edges $u_i v_i$ ($1 \leq i \leq 2^{n-2}$) in Q_n such that $u_i \in S_n$, $v_i \in T_n$ and $A(u_i, v_j) = 0$ for all $i \neq j$. Moreover, these edges can be chosen so that the last binary coordinates of vertices u_i and v_i are identical.

Proof. We proceed by induction on n . For $n = 3$, these edges are $u_1 = (0, 0, 0)$, $v_1 = (1, 0, 0)$, $u_2 = (0, 1, 1)$ and $v_2 = (1, 1, 1)$. Suppose the statement of theorem is true for Q_n . Then by Observation 2.3, $(u_i \times 0)(v_i \times 0)$ would provide 2^{n-2} such edges for Q_{n+1} . Another 2^{n-2} edges of Q_{n+1} can be obtain from $u_i v_i$ by replacing the last coordinates of u_i and v_i with “1, 1” or “0, 1” depending on whether that last coordinate is 0 or 1, respectively. Table 3 illustrates this process from Q_3 to Q_4 . $\square$

Q_3		Q_4	
u_i	v_i	u_i	v_i
(0,0,0)	(1,0,0)	(0,0,0,0) (0,0,1,1)	(1,0,0,0) (1,0,1,1)
(0,1,1)	(1,1,1)	(0,1,1,0) (0,1,0,1)	(1,1,1,0) (1,1,0,1)

Table 3: The induction step of Theorem 2.8, from Q_3 to Q_4 .

3 Some Established Results

In this section we further investigate the elements of cordial sets of hypercubes. For the sake of completeness of this article, we wish to state some of the facts that are already established in [15].

Definition 3.1. A *matching* in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a *perfect matching* if every vertex of G is incident with an edge in M .

Theorem 3.2. Let M be any perfect matching of a graph G . Then in any friendly coloring of G an even number of elements of M are labeled 0.

Proof. We note that every graph with a perfect matching has an even number of vertices. Let $M = \{e_1, e_2, \dots, e_n\}$ with $e_i = u_i v_i$ be a perfect matching of G and $f : V(G) \rightarrow \mathbb{Z}_2$ an arbitrary

friendly coloring of G . If $f_*(e_i) = 1$ for all $i = 1, 2, \dots, n$, then the statement of the theorem is true. Assume, without loss of generality, that $f_*(e_1) = f_*(e_2) = \dots = f_*(e_k) = 0$ ($1 \leq k \leq n$). Then $f_*(e_{k+1}) = \dots = f_*(e_n) = 1$, which implies that $n - k$ vertices of G are labeled 0 and another $n - k$ vertices are labeled 1. These are exactly the vertices that are incident with the edges e_r ($k + 1 \leq r \leq n$). For f to be a friendly coloring, k has to be even, say $k = 2r$, so that r elements of $M = \{e_1, e_2, \dots, e_k\}$ are adjacent to vertices labeled 1 and another r elements are adjacent to vertices labeled 0. $\square$

Lemma 3.3. *The hypercube Q_n is 1-factorable. In fact, it has n disjoint perfect matchings, and each one has 2^{n-1} elements.*

Proof. Note that Q_n is an n -regular bipartite graph. $\square$

Corollary 3.4. *For any binary friendly coloring of Q_n ($n \geq 2$), $e_f(0)$ and $e_f(1)$ are even.*

Corollary 3.5. *For $n \geq 2$, every element of $C(Q_n)$ is a multiple of 4.*

Theorem 3.6. *For any friendly coloring $f : V(Q_n) \rightarrow \mathbb{Z}_2$ that is not maximum (i.e., $N(f) \neq n2^{n-1}$),*

$$N(f) \leq n2^{n-1} - 4(n - 1).$$

Proof. We observe that there are six different ways of coloring the vertices of an square Q_2 by elements of $\mathbb{Z}_2$, as illustrated in Figure 3. In each of these colorings, which are not necessarily friendly, the number of edges labeled 1 (or 0) is even.

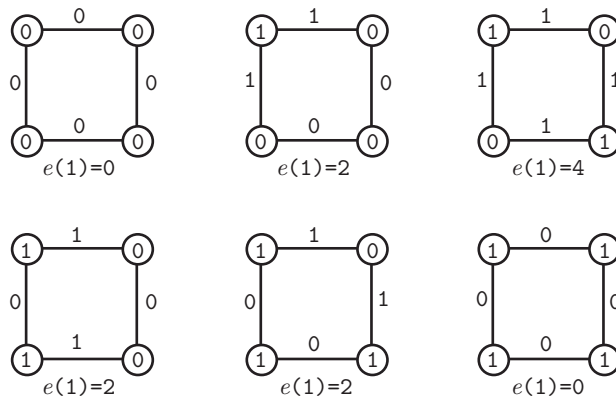


Figure 3: Six possible ways of binary colorings of Q_2 .

Now for any friendly coloring $f : V(Q_n) \rightarrow \mathbb{Z}_2$, its induced edge labeling will label at least one edge by 1. Since $e_f(1)$ is even, we have $e_f(1) \geq 2$. Therefore, there are at least two edges labeled 1. We can assume that these edges are disjoint. These two edges are shared by $n - 1$ squares and an even number of edges of these squares are labeled 1, which implies that $e_f(1) \geq 2(n - 1)$. Therefore, by Equation (1.1), $N(f) = n2^{n-1} - 2e_f(1) \leq n2^{n-1} - 4(n - 1)$. Later, we shall see that there is a friendly coloring that produces the index $n2^{n-1} - 4(n - 1)$. $\square$

4 New Results

First we utilize the notations mentioned in Remark 2.2 and point out that the n -cube is a regular bipartite graph with partite sets S_n and T_n . Let $\varphi : V(Q_n) \rightarrow \mathbb{Z}_2$ be defined by

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in S_n; \\ 1 & \text{if } x \in T_n. \end{cases} \quad (4.1)$$

Then φ is a friendly coloring with $e_\varphi(1) = n2^{n-1}$, $e_\varphi(0) = 0$. Therefore, $N(\varphi) = n2^{n-1}$. This is the *maximum friendly coloring* of Q_n and any other friendly coloring of Q_n is obtained by switching certain number of 0's of S_n with the same exact number of 1's in T_n .

Lemma 4.1. *For any k ($0 \leq k \leq 2^{n-3}$), $n2^{n-1} - 4nk \in C(Q_n)$.*

Proof. Let φ be the maximum friendly coloring of Q_n . By Theorem 2.5, there are 2^{n-3} pair of vertices (u_i, v_i) such that $u_i \in S_n$, $v_i \in T_n$ and $A(u_i, v_j) = 0$ for all i, j . Let us choose k such pairs u_i, v_i ($1 \leq i \leq k$) and define $f_k : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$f_k(x) = \begin{cases} 1 - \varphi(x) & \text{if } x = u_i, v_i \text{ } (1 \leq i \leq k); \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Then f_k is a friendly labeling with $e_{f_k}(0) = 2kn$. Therefore, by Equation (1.1), $N(f_k) = n2^{n-1} - 4kn$. $\square$

An immediate consequence of Lemma 4.1 is the fact that Q_n is cordial.

Corollary 4.2. *For any $n \geq 2$, the hypercube Q_n is cordial.*

Proof. The statement is true for Q_2 , and for $n \geq 3$, we use Lemma 4.1 with $k = 2^{n-3}$. $\square$

Lemma 4.3. *For any k ($0 \leq k \leq 2^{n-2}$), the number $n2^{n-1} - 4k(n - 1)$ is in $C(Q_n)$.*

Proof. Let $\varphi : V(Q_n) \rightarrow \mathbb{Z}_2$ be the maximum friendly coloring of Q_n . By Theorem 2.8, there are 2^{n-2} edges $u_i v_i$ in Q_n such that $u_i \in S_n$, $v_i \in T_n$ and $A(u_i, v_j) = 0$ for all $i \neq j$. Let us choose k such edges $u_i v_i$ ($i = 1, 2, \dots, k$) and define $g_k : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$g_k(x) = \begin{cases} 1 - \varphi(x) & \text{if } x = u_i, v_i \text{ } (i = 1, 2, \dots, k); \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Then g_k is a friendly labeling with $e_{g_k}(0) = 2k(n-2)$.

Therefore, by Equation (1.1), $N(g_k) = n2^{n-1} - 4k(n-1)$. □

The combination of Lemmas 4.1 and 4.3 provide the following useful fact:

Theorem 4.4. *For any x ($0 \leq x \leq 2^{n-3}$) and any y ($0 \leq y \leq 2^{n-2} - 2x$), the numbers*

$$N(x, y) = |n2^{n-1} - 4nx - 4y(n-1)| \quad (4.2)$$

are in $C(Q_n)$.

The advantage of Equation (4.2) is that one can easily evaluate $N(x, y)$ to obtain a fairly good number of cordial indices of $C(Q_n)$. For example, for Q_4 , Equation (4.2) provides all the cordial indices of Q_4 but 12. These numbers are in the upper triangle of Table 4.

$N(x, y)$	$y = 0$	1	2	3	4
$x = 0$	32	20	8	4	16
$x = 1$	16	4	8		
$x = 2$	0				

Table 4: The friendly indices generated by $N(x, y)$ for Q_4 .

In case of Q_5 , Equation (4.2) provides all the cordial indices of Q_5 but 52 and 56, as illustrated in Table 5.

Theorem 4.5. *For $n \geq 6$, in order to obtain all the friendly indices generated by Equation (4.2), it is enough to consider the values of $N(x, y)$ when $0 \leq x + y \leq 2^{n-3}$.*

Proof. By direct examination of Table 6, we see that the statement of the theorem is true for $n = 6$. Following the structure of Table 6, the numbers produced by $N(x, y)$ when $0 \leq x + y \leq 2^{n-3}$ will be referred to as the upper triangular part and the rest of them are referred to as the

$N(x, y)$	$y = 0$	1	2	3	4	5	6	7	8
$x = 0$	80	64	48	32	16	0	16	32	48
$x = 1$	60	44	28	12	4	20	36		
$x = 2$	40	24	8	8	24				
$x = 3$	20	4	12						
$x = 4$	0								

Table 5: The friendly indices generated by $N(x, y)$ for Q_5 .

lower triangular part of $N(x, y)$. Note that when $0 \leq x + y \leq 2^{n-3}$, Equation (4.2) becomes $N(x, y) = n2^{n-1} - 4nx - 4y(n-1)$. Furthermore, on the main diagonal, when $x + y = 2^{n-3}$, the entries are $N(2^{n-3}, 0) = 0$, $N(2^{n-3} - 1, 1) = 4$, $8, \dots, N(0, 2^{n-3}) = 2^{n-1}$. This trend continues on other diagonals parallel to the main diagonal, when $x + y = a$ ($1 \leq a \leq 2^{n-3}$). On the diagonal $x + y = a$, the entries start with $N(a, 0)$ and increase by 4 up to $N(0, a)$. These entries will cover all consecutive multiples of 4, starting from 0 up to $N(0, a - 1)$, as long as $N(0, a) \geq N(a - 1, 0) - 4$. The multiples of 4 that are not covered by the values of $N(x, y)$ occur whenever $N(0, a) < N(a - 1, 0) - 4$, or on the diagonal $x + y = a$ with $1 \leq a \leq n - 2$. Therefore, the first multiple of 4 that is missing in the upper triangular part is $N(n - 3, 0) - 4$.

Also, we note that in the lower triangular part, when $x + y > 2^{n-3}$, Equation (4.2) becomes

$$N(x, y) = -n2^{n-1} + 4nx + 4y(n-1),$$

which is increasing in x and y . Therefore, when $x = a$ is fixed, the maximum value of $N(a, y)$ would be $N(a, 2^{n-2} - 2a)$. In fact, the largest entry of the lower triangular part is $N(0, 2^{n-2})$ and $N(0, 2^{n-2}) < N(n-3) - 4$ or $4n^2 + 4 < 2^n + 12n$ which is true for all $n \geq 7$. This proves that for $n \geq 6$, it is guaranteed that every friendly index supplied by $N(x, y)$ appear in the upper triangular part when $0 \leq x + y \leq 2^{n-3}$. $\square$

Although the number of friendly indices of Q_n increases exponentially, another advantage of Equation (4.2) is that after applying it, the missing multiples of 4 (which are the other possible friendly indices) increases quadratically. The following theorem establishes this fact:

Theorem 4.6. *The number of multiples of 4 that cannot be obtained by Equation (4.2) is $(n-1)(n-2)/2$.*

Proof. As we noticed in the proof of Theorem 4.5, the entries on the main diagonal, where

$x + y = 2^{n-3}$, are $0, 4, 8, \dots, 2^{n-1}$. This trend continues on other diagonals parallel to the main diagonal, when $x + y = a$ ($1 \leq a \leq 2^{n-3}$). On the diagonal $x + y = a$, the entries start with $N(a, 0)$ and increase by 4 up to $N(0, a)$. These entries will cover all consecutive multiples of 4 starting from 0 up to $N(0, a - 1)$, as long as $N(0, a) \geq N(a - 1, 0) - 4$. The multiples of 4 that are not covered by the values of $N(x, y)$ occur whenever $N(0, a) < N(a - 1, 0) - 4$, or on the diagonal $x + y = a$ with $1 \leq a \leq n - 2$. In these situations the $n - a - 1$ missing multiples of 4 are $N(0, a) + 4, N(0, a) + 8, N(0, a) + 12, \dots, N(a - 1, 0) - 4$. Therefore, the function $N(x, y)$ covers all multiples of four between 0 and $n2^{n-1}$ except $\sum_{a=1}^{n-2} (n - a - 1) = (n - 1)(n - 2) - (n - 1)(n - 2)/2 = (n - 1)(n - 2)/2$ of them. $\square$

Corollary 4.7. *The number of possible friendly indices of Q_n that are not obtained by Equation (4.2) is $(n - 2)(n - 3)/2$.*

Proof. By Theorem 3.6, we know that if $N(f)$ is not the maximum friendly index of Q_n , then $N(f) \leq n2^{n-1} - 4n + 4$. This implies that the $n - 2$ multiples of 4 ranging from $n2^{n-1} - 4n + 8$ to $n2^{n-1} - 4$ are not friendly indices of Q_n . Therefore, the actual number of possible friendly indices of Q_n that are not obtained by Equation (4.2) is $(n - 1)(n - 2)/2 - (n - 2) = (n - 2)(n - 3)/2$. $\square$

5 Cordial Sets of Q_n ($n \leq 6$)

Unfortunately, Equation (4.2) does not provide all the friendly indices of Q_n . In the process of its formulation we only switch the labels of vertices $u_i \in S_n$ and $v_i \in T_n$ where either $u_i v_i$ are disjoint edges or they are not adjacent, and this process does not consider certain cases, such as the ones illustrated in Figure 4.



Figure 4: Two examples of the cases that are not considered by Equation (4.2)

In the following lemma, we consider the two cases mentioned in Figure 4 and switch the labels (zeros and ones) of each figure.

$N(x, y)$	$y = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$x = 0$	192	172	152	132	112	92	72	52	32	12	8	28	48	68	88	108	128
$x = 1$	168	148	128	108	88	68	48	28	8	12	32	52	72	92	112		
$x = 2$	144	124	104	84	64	44	24	4	16	36	56	76	96				
$x = 3$	120	100	80	60	40	20	0	20	40	60	80						
$x = 4$	96	76	56	36	16	4	24	44	64								
$x = 5$	72	52	32	12	8	28	48										
$x = 6$	48	28	8	12	32												
$x = 7$	24	4	16														
$x = 8$	0																

Table 6: The friendly indices generated by $N(x, y)$ for Q_6 .

Lemma 5.1. *For any $n \geq 3$,*

- (a) $n2^{n-1} - 8n + 12$ *is a friendly index of Q_n .*
- (b) $n2^{n-1} - 8n + 16$ *is a friendly index of Q_n .*

Proof. For the proof of this lemma, we utilize the notations mentioned in Remark 2.2 and let $\varphi : V(Q_n) \rightarrow \mathbb{Z}_2$ be the maximum friendly coloring of Q_n defined by $\varphi(x) = \begin{cases} 0 & \text{if } x \in S_n; \\ 1 & \text{if } x \in T_n. \end{cases}$

(a) Consider the vertices $u_1 = (0, 0, \dots, 0)$, $v_1 = (1, 0, 0, 0, \dots, 0)$, $u_2 = (1, 1, 0, 0, \dots, 0)$, $v_2 = (1, 1, 1, 0, \dots, 0)$ and define $\alpha : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\alpha(x) = \begin{cases} 1 - \varphi(x) & \text{if } x \in \{u_1, u_2, v_1, v_2\}; \\ \varphi(x) & \text{otherwise,} \end{cases}$$

Then α is a friendly coloring with $e_\alpha(0) = 4n - 6$. Therefore, by Equation (1.1), $N(\alpha) = n2^{n-1} - 2e_\alpha(0) = n2^{n-1} - 8n + 12$.

(b) Consider the vertices $u_1 = (0, 0, \dots, 0)$, $v_1 = (1, 0, 0, 0, \dots, 0)$, $u_2 = (1, 1, 0, 0, \dots, 0)$, $v_2 = (0, 1, 0, 0, \dots, 0)$ which are the vertices of a square and define $\beta : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$\beta(x) = \begin{cases} 1 - \varphi(x) & \text{if } x \in \{u_1, u_2, v_1, v_2\}; \\ \varphi(x) & \text{otherwise,} \end{cases}$$

Then β is a friendly coloring with $e_\beta(0) = 4n - 8$. Therefore, by Equation (1.1), $N(\beta) = n2^{n-1} - 2e_\beta(0) = n2^{n-1} - 8n + 16$. $\square$

For example, for Q_4 , Equation (4.2) supplies all the cordial indices of Q_4 but 12. However, Lemma 5.1 (with $n = 4$) provides this number.

For Q_5 , the equation (4.2), supplies all the cordial indices of Q_5 but 52 and 56. However, Lemma 5.1 (with $n = 5$) provides these two numbers.

For Q_6 , the upper triangular part of Table 6, which resulted from the values of Equation (4.2), supplies all the cordial indices of Q_6 but 116, 136, 140, 156, 160 and 164. Among these six numbers, 156 and 160 are provided by Lemma 5.1 (with $n = 6$).

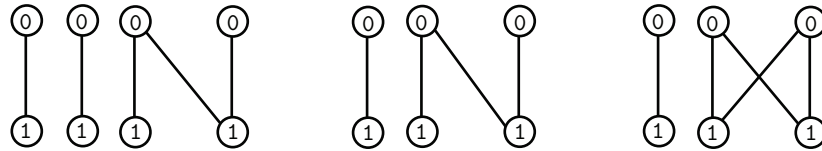


Figure 5: Three more examples of the cases not considered by Equation (4.2)

The numbers 116, 136 and 140 can also be obtained by considering cases mentioned in Figure 5, respectively.

Inspired by Lemma 5.1, we can generalize Equation (4.2) as follows. Let φ be the maximum friendly coloring of Q_n defined by Equation (4.1) and choose r ($1 \leq r \leq 2^{n-2}$) pairs of vertices (u_i, v_i) with $u_i \in S_n$, $v_i \in T_n$. Define $f : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$f(x) = \begin{cases} 1 - \varphi(x) & \text{if } x = u_i, v_i \ (0 \leq i \leq r); \\ \varphi(x) & \text{otherwise,} \end{cases}$$

Then f is a friendly coloring with $K(r) = e_f(0) = 2rn - 2 \sum_{0 \leq i, j \leq r} A(u_i, v_j)$ where $A(u_i, v_j)$ is the adjacency function defined in 2.4. Therefore, by Equation (1.1),

$$N(f) = |n2^{n-1} - 2K(r)|. \quad (5.1)$$

For brevity, we denote $\sum_{0 \leq i, j \leq r} A(u_i, v_j)$ by $A(r)$. Hence, $K(r) = e_f(0) = 2rn - 2A(r)$.

Observation 5.2. Note that after we choose the r pair of vertices (u_i, v_j) with $u_i \in S_n$ and $v_i \in T_n$, then $A(r)$ counts the number of possible $u_i v_j$ edges. Also, we observe that if $r = 2^k$, then the maximum value of $A(2^k)$ occurs when the subgraph induced by these 2^k selected pair of vertices form Q_{k+1} . Therefore, $A(2^k) \leq (k+1)2^k$.

For example, $A(16) \leq 80$, and the right hand side is the number of edges of Q_5 . Now if we remove a pair of vertices, since every vertex of Q_5 has degree 5, then a total of 9 or 10 edges are eliminated, depending on whether the pair of vertices are adjacent or not. This means $A(15) \leq 71$. Another removal of a pair of vertices results in elimination of at least 7 other edges, that is, $A(14) \leq 64$. Continuation of this process provides upper bounds for $A(r)$. Clearly, $A(2) \leq 4$, $A(3) \leq 7$ and $A(4) \leq 12$. Other upper bounds are listed in Table 7.

r	5	6	7	8	9	10	11	12	13	14	15	16
$A(r)$	15	20	25	32	39	44	49	52	57	64	71	80

Table 7: The upper bounds of $A(r)$ for different values of r .

In case of Q_6 and the number 164, we have the following fact.

Lemma 5.3. *The number 164 is not a friendly index of Q_6 .*

Proof. Assume to the contrary that $164 \in C(Q_6)$. Then by Equation (5.1), either $K(r) = 14$ or $K(r) = 178$ for some $0 \leq r \leq 16$. Since $n = 6$, then $K(r) = 2nr - 2A(r) = 12r - 2A(r)$.

The case of $K(r) = 14$ requires that $A(r) = 6r - 7$ for some $2 \leq r \leq 16$. However, this is not feasible, because the values of $A(r) = 6r - 7$ exceed the upper bounds of $A(r)$ listed in the second row of Table 7.

Suppose $K(r) = 178$ or $12r - 2A(r) = 178$. Since $A(r) \geq 0$, then either $r = 15$ or 16 which implies that either $A(15) = 1$ or $A(16) = 7$, respectively.

Now, we observe that after we choose the r pair of vertices (u_i, v_j) with $u_i \in S_n$ and $v_i \in T_n$, then $A(r)$ counts the number of possible $u_i v_j$ edges. Clearly, $A(32) = 192$. When we remove

one pair of vertices (u, v) , then depending on whether uv is an edge or not, we eliminate 11 or 12 edges, respectively. Therefore, $A(31) \geq 180$. Similarly, $A(30) \geq 168$, $A(29) \geq 156, \dots$ and $A(21) \geq 60$.

Also, $A(j) \geq 0$ ($0 \leq j \leq 10$). Other lower bounds of $A(r)$ are listed in Table 8.

r	10	11	12	13	14	15	16	17	18	19	20	21
	0	0	0	1	1	2	8	18	28	38	48	60

Table 8: The lower bounds of $A(r)$ for different values of r .

This implies that $A(15) = 1$ and $A(16) = 7$ are not possible. Therefore, 164 is not a friendly index of Q_6 . $\square$

In summary, we just proved that $C(Q_6) = \{4k : 0 \leq k \leq 40\} \cup \{168, 172, 192\}$.

Finally, based on our observations in other cases, we would like to improve the conjecture that is stated in [15].

Conjecture 5.4. For $n \geq 3$,

$$C(Q_n) = \{4k : 0 \leq k \leq n2^{n-3} - 2n + 4\} \cup \{n2^{n-1} - 4n, n2^{n-1} - 4n + 4, n2^{n-1}\}.$$

References

- [1] I. Cahit, Cordial Graphs: a weaker version of graceful and harmonious graphs, *Ars Combinatoria* **23** (1987), 201-207.
- [2] N. Cairnie and K. Edwards, The Computational Complexity of Cordial and Equitable Labelling, *Discrete Mathematics*, **216** (2000), 29-34.
- [3] G. Chartrand, S-M Lee and P. Zhang, Uniformly Cordial Graphs, *Discrete Mathematics*, **306** (2006), 726-737.
- [4] G. Chartrand and P. Zhang, A First Course in Graph Theory, *Dover Publications, New York* (2012).
- [5] Joseph A. Gallian, A dynamic survey of graph labeling, *Electronic Journal of Combinatorics*, (2014).
- [6] M. Hovay, A-cordial Graphs, *Discrete Mathematics* **93** (1991), 183-194.
- [7] W.W. Kirchherr, On the Cordiality of Certain Specific Graphs, *Ars Combinatoria* **31** (1991), 127-138.
- [8] W.W. Kirchherr, NEPS operations on cordial graphs, *Discrete Mathematics*, **115** (1993), 201-209.
- [9] Y.H. Lee, H.M. Lee, and G.J. Chang, Cordial Labelings of Graphs, *Chinese J. Math.* **20** (1992), 263-273.

- [10] S-M. Lee and A. Liu, A Construction of Cordial Graphs from Smaller Cordial Graphs, *Ars Combinatoria* **32** (1991), 209-214.
- [11] E. Salehi and D. Bayot, Friendly Index Sets of Grids, *Utilitas Mathematica* **81** (2010), 121-130.
- [12] E. Salehi and D. Corral, Fully Cordial Trees, *Journal of Combinatorial Mathematics and Combinatorial Computing* **98** (2016), 171-183.
- [13] E. Salehi and Shipra De, On a Conjecture Concerning the Friendly Index Sets of Trees, *Ars Combinatoria* **90** (2009), 371-381.
- [14] E. Salehi and S-M. Lee, On Friendly Index Sets of Trees, *Congressus Numerantium* **178** (2006), 173-183.
- [15] E. Salehi, Y. Mukhin and S. W. Saputro, Cordial Sets of Hypercubes, *Bulletin of Institute of Combinatorics and its Applications* **75** (2015), 95-106.