

On Cordial Sets of Regular Bipartite Graphs

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Abstract

For a graph $G = (V, E)$, a binary labeling (coloring) $f : V(G) \rightarrow \mathbb{Z}_2$, is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. The friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f_* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f_*(xy) = |f(x) - f(y)|$ for all $xy \in E(G)$. Let $e_f(i) = |f_*^{-1}(i)|$ be the number of edges labeled i . The friendly index set (or cordial set) of the graph G , denoted by $C(G)$, is defined by

$$C(G) = \{|e_f(1) - e_f(0)| : f \text{ is a friendly vertex labeling of } G\}.$$

In this paper we consider some classes of regular bipartite graphs and completely determine their friendly index sets.

Key Words: Friendly labeling; cordial graphs; friendly index set; index function.

AMS Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [4]. Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}_2$ be a binary vertex labeling (or coloring) of G . For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices colored 0 is almost the same as the number of vertices colored 1. Any friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f_* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f_*(xy) = |f(x) - f(y)|$ for all $xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f_*^{-1}(i)|$ be the number of edges of G that are labeled i . The number $N(f) = |e_f(1) - e_f(0)|$ is called the *cordial index* of f . A graph G is said to be *cordial* if it admits a friendly labeling with cordial index 0 or 1.

To illustrate the above concepts, consider the graph G of Figure 1, which has six vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that three vertices be labeled 0 and the other three 1.

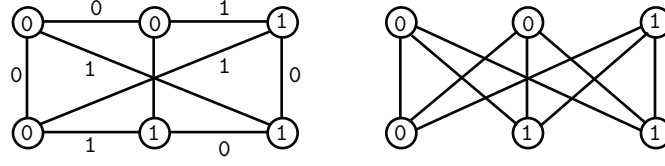


Figure 1: An Example of friendly labeling with index 1.

Figure 1 also shows the associated edge labeling of G , where five edges have labels 1 while the other four edges have labels 0. Therefore, the cordial index of this labeling is $5 - 4 = 1$.

In 1978, I. Cahit [1] introduced the concept of cordial labeling as a weakened version of the less tractable graceful and harmonious labeling. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. Hovay [7], later generalized the concept of cordial graphs and introduced A -cordial labelings, where A is an abelian group. A graph G is said to be A -cordial if it admits a labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cairnie-Edwards [2] proved that the problem of deciding whether or not a graph G is cordial is NP-complete, as conjectured by Kirchherr [9]. Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [5, 8, 10, 11, 13, 14, 15, 16].

In [3] Chartrand-Lee-Zhang introduced the concept of *friendly index set* of a graph G . The friendly index set or *cordial set* of the graph G , denoted by $C(G)$, is defined by

$$C(G) = \{N(f) : f \text{ is a friendly labeling of } G\}.$$

Note that if 0 or 1 is in $C(G)$, then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality.

For the graph G in Figure 1, it is easy to verify that $C(G) = \{1, 9\}$. The friendly colorings of G that provide the other cordial index is presented in Figure 2.

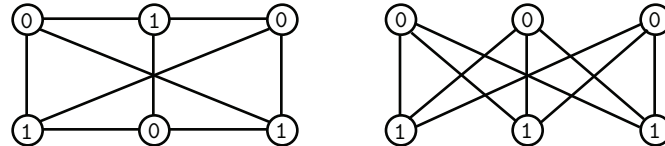


Figure 2: A friendly labeling of G with cordial index 9.

In this paper, we focus on the group $A = \mathbb{Z}_2$ and, among other things, determine the friendly index sets of some regular bipartite graphs. A friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ is called a *maximum*

friendly labeling of G if its friendly index is maximal, that is, $N(f) = |E(G)|$. In this case, we call $N(f)$ the maximum friendly index of G . Also, if $f : V(G) \rightarrow \mathbb{Z}_2$ is a friendly labeling, so is its inverse labeling $g : V(G) \rightarrow \mathbb{Z}_2$ defined by $g(v) = 1 - f(v) \forall v \in V(G)$. Moreover, $N(g) = N(f)$.

For a graph $G = (p, q)$ of size q , and a friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of G , we have

$$N(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)| = |q - 2e_f(0)|. \quad (1.1)$$

Therefore, to find the cordial index of f it is enough to find $e_f(1)$ or $e_f(0)$. Moreover, to determine the cordial set of G it is enough to compute $e_f(1)$ [or $e_f(0)$] for different friendly labelings.

Another immediate consequence of Equation (1.1) is the following useful fact:

Observation 1.1. *For a graph G of size q , $C(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.*

2 Regular Bipartite graphs

In this section, some of the properties of the Regular Bipartite Graph (RBG) that are utilized for finding its cordial set are investigated.

Definition 2.1. A *matching* in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a *perfect matching* if every vertex of G is incident with an edge in M .

The following theorem is established in [18]. For the sake of completeness of this article, we also mention it here:

Theorem 2.2. *Let M be any perfect matching of a graph G . Then in any friendly coloring of G an even number of elements of M are labeled 0.*

Proof. We note that every graph with a perfect matching has an even number of vertices. Let $M = \{e_1, e_2, \dots, e_n\}$ with $e_i = u_i v_i$ be a perfect matching of G and $f : V(G) \rightarrow \mathbb{Z}_2$ an arbitrary friendly coloring of G . If $f_*(e_i) = 1$ for all $i = 1, 2, \dots, n$, then the statement of the theorem is true. Assume, without loss of generality, that $f_*(e_1) = f_*(e_2) = \dots = f_*(e_k) = 0$ ($1 \leq k \leq n$). Then $f_*(e_{k+1}) = \dots = f_*(e_n) = 1$, which implies that $n - k$ vertices of G are labeled 0 and another $n - k$ vertices are labeled 1. These are exactly the vertices that are incident with the edges e_r ($k + 1 \leq r \leq n$). For f to be a friendly coloring, k has to be even, say $k = 2r$, so that r elements of $M = \{e_1, e_2, \dots, e_k\}$ be adjacent to vertices labeled 1 and another r elements be adjacent to vertices labeled 0. $\square$

Corollary 2.3. *For any binary friendly coloring of an r -regular bipartite graph G with $|E(G)| = q$, $e_f(0)$ is even. Furthermore,*

$$C(G) \subseteq \{|q - 4k| : 0 \leq k \leq \lfloor q/2 \rfloor\} \quad (2.1)$$

Proof. Note that G is 1-factorable and has r mutually disjoint perfect matchings $M_1, M_2 \dots M_r$. For each i the number of elements of M_i that are labeled 0 is even. Hence $e_f(0)$ is even and Relation (2.1) follows from Equation (1.1). $\square$

Lemma 2.4. *Let C_{2n} be an even cycle. Then for any binary vertex coloring $f : C_{2n} \rightarrow \mathbb{Z}_2$, not necessarily friendly, an even number of edges are labeled 1 by the induced edge labeling of f .*

Proof. If $f(v) = 0$ (or 1) for all $v \in V(G)$, then $e_f(1) = 0$ is even. Assume that there are vertices that are labeled 0 as well as vertices that are labeled 1. Choose a vertex that is labeled 0 and navigate the cycle clockwise. Note that anytime we encounter a block of vertices that are labeled 1, this block of vertices are followed by a vertex that is labeled 0. This justifies the fact that $e_f(1)$ is even. $\square$

Definition 2.5. *Let $G = (p, q)$ be a regular bipartite graph with partite sets S and T . Consider the function $\varphi : V(G) \rightarrow \mathbb{Z}_2$ defined by*

$$\varphi(x) = \begin{cases} 0 & \text{if } x \in S; \\ 1 & \text{if } x \in T. \end{cases} \quad (2.2)$$

Then φ is a friendly coloring with $e_\varphi(1) = q$ and $e_\varphi(0) = 0$. Therefore, $N(\varphi) = q$. This is the maximum friendly coloring of G .

Remark 2.6. *We observe that when $G = (p, q)$ is a regular bipartite graph, then q , the number of edges of G , is a friendly index. Any other friendly index of G is obtained by switching certain number of 0's of S with the same exact number of 1's in T . Now depending on whether we switch the labels of edges or pair of non-adjacent vertices, we get different indices. As a result, it is extremely crucial to identify independent edges $u_i v_i$ and pair of non-adjacent vertices (u_i, v_i) such that $u_i \in S$, $v_i \in T$.*

Lemma 2.7. *Let $G = (p, q)$ be an r -regular bipartite graph. If f is not the maximum friendly coloring of G , then $e_f(0) \geq 2(r - 1)$.*

Proof. Let φ be the maximum friendly coloring of G defined by Equation (2.6). For this function $e_\varphi(0) = 0$ and every edge is labeled 1. The next smallest available number for $e(0)$ is obtained when we switch the labels of vertices of exactly one edge. In doing so, we obtain another friendly coloring with $e(0) = 2(r - 1)$. $\square$

Definition 2.8. For a graph G , we define the adjacency function $A : V(G) \times V(G) \rightarrow \mathbb{Z}_2$ by

$$A(u_i, v_j) = \begin{cases} 1 & \text{if } u_i v_j \in E(G) \\ 0 & \text{if } u_i v_j \notin E(G). \end{cases}$$

Lemma 2.9. In the ladder $P_m \times P_2$ there are at least $\lfloor m/2 \rfloor$ edges $u_i v_i$ such that $u_i \in S$, $v_i \in T$ and $A(u_i, v_j) = 0$ for all $i \neq j$. Here S and T are the partite sets of $P_m \times P_2$.

Proof. Let $x_1 \sim x_2 \sim \dots \sim x_m \sim x_1$ be the vertices of P_m and $V(P_2) = \{a, b\}$. Also, let H be the subgraph of $P_m \times P_2$ induced by the vertices $\{u_i = (x_i, a) : 1 \leq i \leq m\}$ and K be the subgraph induced by the vertices $\{v_i = (x_i, b) : 1 \leq i \leq m\}$. These two subgraphs are isomorphic with P_m and $P_m \times P_2$ is the disjoint union of H and K with the added edges $u_i v_i$ ($1 \leq i \leq m$). These m edges $u_i v_i$ form a perfect matching of $P_m \times P_2$ and can be thought of as the “steps of the ladder”. If we choose every other one of these steps $u_{2k-1} v_{2k-1}$ ($1 \leq k \leq \lfloor m/2 \rfloor$), we get $\lfloor m/2 \rfloor$ desired edges. $\square$

Lemma 2.10. In the ladder $P_m \times P_2$ there are at least $\lfloor m/3 \rfloor$ pair of vertices (c_i, d_j) such that $c_i \in S$, $d_j \in T$ and $A(c_i, d_j) = 0$ for all i, j .

Proof. Using the notation of Lemma 2.9, we observe that every consecutive three steps of $P_m \times P_2$, such as $u_1 v_1$, $u_2 v_2$, $u_3 v_3$ provides one such pair of vertices; namely, (u_1, v_3) . Therefore, (u_{3k-2}, v_{3k}) , when $k = 1, 2, \dots, \lfloor m/3 \rfloor$ form $\lfloor m/3 \rfloor$ pair of non-adjacent vertices. $\square$

Remark 2.11. The statement of Lemma 2.10 can be improved. In fact, one can easily show that there are at least $\lfloor 2m/5 \rfloor$ pair of vertices (c_i, d_j) such that $c_i \in S$, $d_j \in T$ and $A(c_i, d_j) = 0$ for all i, j . To achieve this improvement, we simply observe that every five steps provide a couple of such pairs, as it is shown in Figure 3. In this figure, the dashed curves are identifying the pairs of non-adjacent vertices.

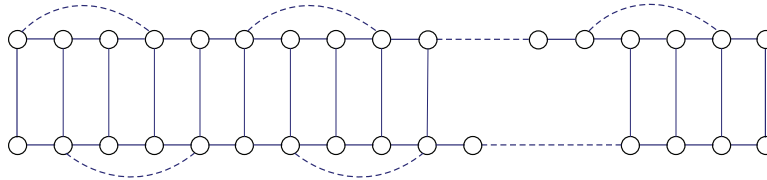


Figure 3: The process of selection of pair of vertices for Remark 2.11.

3 Cordial Sets of Prisms $C_{2n} \times P_2$

The prism $P(2n) = C_{2n} \times P_2$ is a cubic bipartite graph of order $4n$, has size $6n$ and its cordial set is a subset of $\{6n, 6n - 8, 6n - 12, 6n - 16, \dots, 6n - 4\lfloor \frac{6n}{4} \rfloor\}$. Let $x_1 \sim x_2 \sim \dots \sim x_{2n} \sim x_1$ be the

vertices of C_{2n} and $V(P_2) = \{a, b\}$. Also, let H be the subgraph of $P(2n)$ induced by the vertices $\{u_i = (x_i, a) : 1 \leq i \leq 2n\}$ and K be the subgraph induced by the vertices $\{v_i = (x_i, b) : 1 \leq i \leq 2n\}$. These two subgraphs are isomorphic with C_{2n} and $P(2n)$ is the disjoint union of H and K with the added edges $u_i v_i$ ($1 \leq i \leq 2n$). These $2n$ edges $u_i v_i$ form a perfect matching of $P(2n)$.

Lemma 3.1. There are n disjoint edges $u_k v_k$ in $P(2n)$ such that $u_k \in S$, $v_k \in T$ and $A(u_i, v_j) = 0$ for all $i \neq j$, where S and T are the two partite sets of $P(2n)$.

Proof. From the $2n$ edges $u_i v_i$ choose every other one $u_{2k-1} v_{2k-1}$ ($k = 1, 2, \dots, n$). $\square$

Lemma 3.2. For any k ($0 \leq k \leq n$), the number $|6n - 8k|$ is a friendly index of $P(2n)$.

Proof. Let φ be the maximum friendly coloring of $P(2n)$. By Lemma 3.1, there are n edges $u_i v_i$ in $P(2n)$ such that $u_i \in S$, $v_i \in T$ and $A(u_i, v_j) = 0$ for all $i \neq j$. Let us choose k such edges $u_i v_i$ ($i = 1, 2, \dots, k$) and define $g_k : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$g_k(x) = \begin{cases} 1 - \varphi(x) & \text{if } x = u_i, v_i \text{ } (i = 1, 2, \dots, k); \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Then g_k is a friendly labeling with $e_{g_k}(0) = 4k$ and, by Equation (1.1), $N(g_k) = |6n - 8k|$. $\square$

Lemma 3.3. There are at least $\lfloor (2n - 1)/3 \rfloor$ pair of vertices (u_i, v_j) in prism $P(2n)$ such that $u_i \in S$, $v_j \in T$ and $A(u_i, v_j) = 0$.

Proof. Note that $P(2n) - \{u_{2n}, v_{2n}\} = P_{2n-1} \times P_2$. Therefore, by Lemma 2.9, there are at least $\lfloor (2n - 1)/3 \rfloor$ pair of non-adjacent vertices of $P_{2n-1} \times P_2$ with the stated properties. These vertices also work for $P(2n)$. $\square$

Lemma 3.4. For any k ($0 \leq k \leq \lfloor (2n - 1)/3 \rfloor$), the number $|6n - 12k|$ is a friendly index of $P(2n)$.

Proof. Let φ be the maximum friendly coloring of $P(2n)$. By Lemma 3.3, there are $\lfloor (2n - 1)/3 \rfloor$ pair of vertices (u_i, v_i) such that $u_i \in S$, $v_i \in T$ and $A(u_i, v_j) = 0$ for all i, j . Let us choose k such pairs u_i, v_i ($i = 1, 2, \dots, k$) and define $f_k : V(Q_n) \rightarrow \mathbb{Z}_2$ by

$$f_k(x) = \begin{cases} 1 - \varphi(x) & \text{if } x = u_i, v_i \text{ } (i = 1, 2, \dots, k); \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Then f_k is a friendly labeling with $e_{f_k}(0) = 6k$ and by Equation (1.1), $N(f_k) = |6n - 12k|$. $\square$

The combination of Lemmas 3.2 and 3.4 provide the following useful fact:

Theorem 3.5. For any x ($0 \leq x \leq \lfloor (2n - 1)/3 \rfloor$) and any y ($0 \leq y \leq n - 2x$), the number

$$N(x, y) = |6n - 12x - 8y| \tag{3.1}$$

is a friendly index of $P(2n)$.

We call $N(x, y)$ cordial index function (or simply index function). The advantage of Equation (3.1) is that one can easily evaluate $N(x, y)$ to obtain all the cordial indices of $P(2n)$. For example, in the case of $P(14)$, Equation (3.1) provides all the cordial indices of $P(14)$. These numbers are arranged in Table 1. Note that in this case there are 7 independent edges $u_i v_i$ as described in Lemma 3.1 and there are 4 pairs of vertices (u_i, v_j) as stated in Lemma 3.3. Also, Equation (3.1) becomes $N(x, y) = |42 - 12x - 8y|$.

$N(x, y)$	$y = 0$	1	2	3	4	5	6	7
$x = 0$	42	34	26	18	10	2	6	14
$x = 1$	30	22	14	6	2	10		
$x = 2$	18	10	2	6				
$x = 3$	6	2						

Table 1: The friendly indices of $P(14)$ generated by $N(x, y)$.

For $P(16)$, Equation (3.1) provides all of its cordial indices. Note that in this case there are 8 independent edges $u_i v_i$ as described in Lemma 3.1 and there are 4 pairs of vertices (u_i, v_j) as stated in Lemma 3.3. Also, Equation (3.1) becomes $N(x, y) = |48 - 12x - 8y|$. These numbers are arranged in Table 2. We would like to mention that the friendly index sets of prisms are known [12]. However, as an application of the index function, we wish to present it here with a new approach.

$N(x, y)$	$y = 0$	1	2	3	4	5	6	7	8
$x = 0$	48	40	32	24	16	8	0	8	16
$x = 1$	36	28	20	12	4	4	12		
$x = 2$	24	16	8	0	8				
$x = 3$	12	4	4						
$x = 4$	0								

Table 2: The friendly indices of $P(16)$ generated by $N(x, y)$.

Theorem 3.6. For all n , $C(P(2n)) = \{6n\} \cup \{6n - 4k : 2 \leq k \leq n + \lfloor n/2 \rfloor\}$.

Proof. The result can easily be checked for $n = 1, 2$. As an example, for $n = 3$, $P(6) = C_6 \times P_2$. In this case $x = 0, 1$ and $0 \leq y \leq 3 - 2x$ and Equation (3.1) provides all the cordial indices of $P(6)$. These numbers are arranged in Table 3.

Let $n \geq 4$. We consider the following two cases:

- (A) Let $n = 2k$. In this case, the index function (3.1) becomes $N(x, y) = |12k - 12x - 8y|$, where $0 \leq x \leq k$ and $0 \leq y \leq 2k - 2x$. If we arrange the values of this index function in a

$N(x, y)$	$y = 0$	1	2	3
$x = 0$	18	10	2	6
$x = 1$	6	2		

Table 3: The friendly indices of $P(6)$ generated by $N(x, y)$.

table, as is done in Table 2, we observe that on diameter $x + y = k$ the entries start from $N(k, 0) = 0$ and increase by 4 up to $N(0, k) = 4k$. This trend continues on the other lines $x + y = a$ ($1 \leq a \leq k$). Since $N(0, a) + 4 \geq N(a - 1, 0)$ is true for all values of $a \geq 2$, the index function (3.1) produces all cordial indices of $P(2n)$ when $0 \leq x + y \leq k$.

- (B) Let $n = 2k + 1$. In this case, the index function (3.1) becomes $N(x, y) = |12k + 6 - 12x - 8y|$, where $0 \leq x \leq k$ and $0 \leq y \leq 2k + 1 - 2x$. With the same argument as in case (A), if we arrange the values of the index function in a table, we notice that all the cordial indices of $P(2n)$ are produced in the upper triangular part of the table except for 2. However, this index can be obtained by considering $N(k, 1) = 2$ or $N(k - 2, 4) = 2$.

This proves the theorem. In the process of the proof we noticed that in order to find the cordial set of $P(2n)$, it is enough to consider the values of $N(x, y)$ when $0 \leq x + y \leq \lfloor n/2 \rfloor$. $\square$

4 Cordial Sets of $C_{2m} \times C_{2n}$

One of the natural extension of $C_{2n} \times P_2$ is the graph $C_{2m} \times C_{2n}$ ($m \geq n \geq 2$), which is called a torus grid. In this section, we completely determine the cordial set of $C_{2m} \times C_{2n}$. The smallest such graph is the 4-cube $Q_4 = C_4 \times C_4$ and its cordial set is known [18] to be $\{0, 4, 8, 12, 16, 20, 32\}$. Therefore, from this point on, we assume that $m > 2$.

Note that $C_{2m} \times C_{2n}$ is a 4-regular bipartite graph of order $4mn$ with $8mn$ edges. As before, we denote the two partite sets by S and T . Moreover, this graph is 1-factorable. That is, the edges can be partitioned by 4 perfect matchings, each one containing $2mn$ edges. By Corollary 2.3, for any friendly coloring of this graph, $e(0)$ and $e(1)$ are even. Therefore,

$$C(C_{2m} \times C_{2n}) \subseteq \{8mn - 4i : 0 \leq i \leq 2mn\}. \quad (4.1)$$

Lemma 4.1. *Let f be any friendly coloring that is not maximum. Then $N(f) \leq 8mn - 12$.*

Proof. Clearly $e_f(1) \neq 0$. By Lemma 2.4, $e_f(1) \geq 2$. However, each edge labeled 1 produces 3 more edges with label 1 in the adjacent squares. This implies that $e_f(1) \geq 6$. Therefore, $N(f) = 8mn - 2e_f(1) \leq 8mn - 12$. $\square$

Lemma 4.2. *The graph $C_{2m} \times C_{2n}$ contains mn independent edges $u_i v_i$ such that $u_i \in T$, $v_i \in S$ and $A(u_i, v_j) = 0$ for all $i \neq j$. Also, this graph contains $\lfloor mn/2 \rfloor$ pair of vertices (u_i, v_j) such that $u_i \in T$, $v_j \in S$ and $A(u_i, v_j) = 0$ for all i, j .*

Proof. Let $x_1 \sim x_2 \sim \dots \sim x_{2n} \sim x_{2n+1} = x_1$ be the vertices of C_{2n} . The graph $C_{2m} \times C_{2n}$ contains $2n$ copies of $C_{2m} \times P_2$. Namely, $C_{2m} \times \{x_i, x_{i+1}\}$ ($1 \leq i \leq 2n$). By Lemma 3.1, each one of these copies contains m independent edges $u_i v_i$ with $u_i \in T$, $v_i \in S$ and $A(u_i, v_j) = 0 \forall i \neq j$. We choose these edges from every other copies of $C_{2m} \times \{x_i, x_{i+1}\}$ and we get mn independent edges with the desired properties. Now, from each pair of mn edges obtained, such as $u_i v_i$ and $u_j v_j$ ($i \neq j$), we choose the pair of vertices (u_i, v_j) . This will provide $\lfloor mn/2 \rfloor$ pair of vertices (u_i, v_j) such that $u_i \in T$, $v_j \in S$ and $A(u_i, v_j) = 0$ for all i, j . $\square$

Theorem 4.3. *For every $0 \leq x \leq mn/2$ and $0 \leq y \leq mn - 2x$ the number*

$$N(x, y) = |8mn - 16x - 12y| \quad (4.2)$$

is a cordial index of $C_{2m} \times C_{2n}$.

Proof. Let φ be the maximum friendly coloring of $C_{2m} \times C_{2n}$ defined by (2.6). We choose x pairs of vertices and y edges as described in Lemma 4.2 and switch their labels. This produces a friendly coloring of $C_{2m} \times C_{2n}$ in which $e(0) = 8x + 6y$. Therefore, the corresponding cordial index would be $|8mn - 2e(0)| = |8mn - 16x - 12y|$. $\square$

As before, we call $N(x, y) = |8mn - 16x - 12y|$ the index function of $C_{2m} \times C_{2n}$. The beauty of this function is that it produces almost all the friendly indices of $C_{2m} \times C_{2n}$. It starts with the maximum friendly index, when the adjacent vertices are labeled differently, and then switches the labels of the edges and pairs of vertices that are described in Lemma 4.2. However, this function does not produce the cordial indices of other cases. One such case is illustrated in Figure 4.

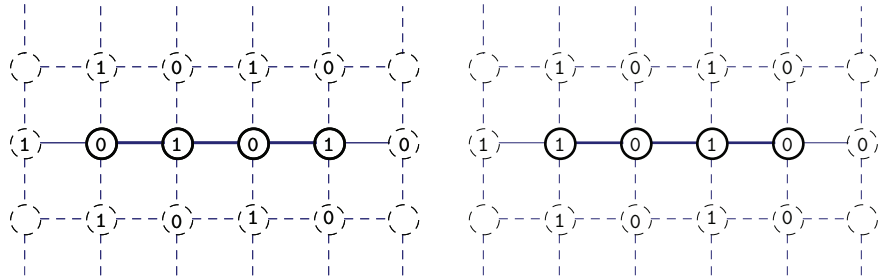


Figure 4: An Example of a case that is not considered by Equation (4.2)

For this particular case, we have the following fact:

Lemma 4.4. *The number $8mn - 20$ is a friendly index of $C_{2m} \times C_{2n}$.*

Proof. Let φ be the maximum friendly coloring of $C_{2m} \times C_{2n}$ and consider four vertices as shown in Figure 4. If we switch the labels of these four vertices, we get another friendly coloring with $e(0) = 10$. Therefore, the corresponding cordial index of this new friendly coloring would be $8mn - 2e(0) = 8mn - 20$. $\square$

Theorem 4.5. $C(C_{2m} \times C_{2n}) = \{8mn\} \cup \{8mn - 4i : 3 \leq i \leq 2mn\}$.

Proof. We consider the following two cases:

- (A) Assume that m or n is even. In this case, $\lfloor mn/2 \rfloor = mn/2$. If we arrange the values of this index function in a table, as is done in Table 4, we observe that on diameter $x + y = mn/2$ the entries start from $N(mn/2, 0) = 0$ and increase by 4 up to $N(0, mn/2) = 2mn$. This trend continues on other lines $x + y = a$ ($2 \leq a \leq mn/2$). Since $N(0, a) + 4 \geq N(a - 1, 0)$ is true for all values of $a \geq 3$, the index function (4.2) produces all multiples of 4 starting from 0 up to $N(0, 2) = 8mn - 24$ for $x + y = a$ ($a \geq 2$), all of which are indices of $C_{2m} \times C_{2n}$. Also, we observe that $N(0, 0) = 8mn$, $N(0, 1) = 8mn - 12$ and $N(1, 0) = 8mn - 16$. The only multiples of 4 that are remaining would be $8mn - 20$, $8mn - 8$ and $8mn - 4$. For these numbers, Lemma 4.4 provides $8mn - 20$ as a cordial index of $C_{2m} \times C_{2n}$ and Lemma 4.1 states that $8mn - 4$ and $8mn - 8$ are not cordial indices of $C_{2m} \times C_{2n}$.

$N(x, y)$	$y = 0$	1	2	3	4	5
$x = 0$	80	68	56	44	32	20
$x = 1$	64	52	40	28	16	
$x = 2$	48	36	24	12		
$x = 3$	32	20	8			
$x = 4$	16	4				
$x = 5$	0					

Table 4: The friendly indices of $C_{10} \times C_4$ generated by Equation (4.2).

- (B) Assume that m and n are both odd numbers. In this case, $\lfloor mn/2 \rfloor = (mn - 1)/2$. If we arrange the values of this index function in a table, as is done in Table 5, we observe that on diameter $x + y = \lfloor mn/2 \rfloor$ the entries start from 8 and increase by 4 up to $2mn + 6$.

This trend continues on other lines $x + y = a$ ($2 \leq a \leq mn/2$). Since $N(0, a) + 4 \geq N(a - 1, 0)$ is true for all values of $a \geq 3$, the index function (4.2) produces all multiples of 4 starting from 8 up to $N(0, 2) = 8mn - 24$ for $x + y = a$ ($a \geq 2$), all of which are indices of $C_{2m} \times C_{2n}$. Also, we

$N(x, y)$	$y = 0$	1	2	3	4
$x = 0$	72	60	48	36	24
$x = 1$	56	44	32	20	
$x = 2$	40	28	16		
$x = 3$	24	12			
$x = 4$	8				

Table 5: The friendly indices of $C_6 \times C_6$ generated by Equation (4.2).

observe that $N(0, 0) = 8mn$, $N(0, 1) = 8mn - 12$ and $N(1, 0) = 8mn - 16$. The only multiples of 4 that are remaining would be 0, 4, $8mn - 20$, $8mn - 8$ and $8mn - 4$. As we noticed in previous case, it is enough to show that 0 and 4 are friendly indices of $C_{2m} \times C_{2n}$. Fortunately, these two indices are also provided by the index function $N((mn - 3)/2, 2) = 0$, $N((mn - 5)/2, 3) = 4$.

□

5 Cordial Sets of Möbius Ladder M_{4n+2}

Möbius Ladder, denoted by M_{2n} , is the cycle C_{2n} with n additional edges joining the diagonally opposite vertices. Equivalently, it is the ladder $P_n \times P_2$ with extra two edges joining the opposite end points of the two copies of P_n . Note that M_{4n} is not bipartite because it contains an odd cycle. However, M_{4n+2} is a bipartite graph.

Let S and T be the two partite sets of M_{4n+2} , $x_1 \sim x_2 \sim \dots \sim x_{2n+1}$ be the vertices of P_{2n+1} and $V(P_2) = \{a, b\}$. Also, let H be the subgraph of M_{2n+2} induced by the vertices $\{u_i = (x_i, a) : 1 \leq i \leq 2n + 1\}$ and K be the subgraph induced by the vertices $\{v_i = (x_i, b) : 1 \leq i \leq 2n + 1\}$. These two subgraphs are isomorphic with P_{2n+1} and M_{4n+2} is the disjoint union of H and K with $2n + 1$ steps $u_i v_i$ ($1 \leq i \leq n$) and two extra edges $u_1 v_{2n+1}$ and $u_{2n+1} v_1$.

In this section we wish to utilize the index function to determine the cordial sets of M_{4n+2}

Lemma 5.1. *There are n independent edges $u_i v_i$ in M_{4n+2} such that $u_i \in S$, $v_i \in T$ and $A(u_i, v_j) = 0$ for all $i \neq j$.*

Proof. Note that $M_{4n+2} - \{u_{2n+1} v_{2n+1}\}$ is the ladder $P_{2n} \times P_2$ with $2n$ steps. The result follows from Lemma 2.9. □

Lemma 5.2. *There are $\lfloor 4n/5 \rfloor$ pair of vertices (c_i, d_i) in M_{4n+2} such that $c_i \in S$, $d_i \in T$ and $A(c_i, d_j) = 0$ for all i, j .*

Proof. Again, we observe that $M_{4n+2} - \{u_{2n+1}v_{2n+1}\}$ is the ladder $P_{2n} \times P_2$ and the results follow from Remark 2.11. $\square$

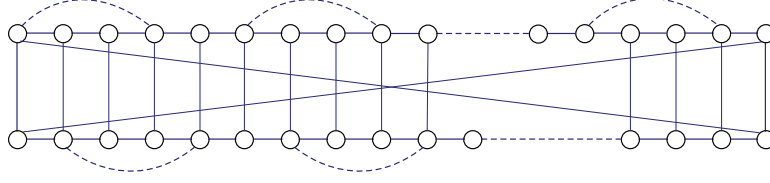


Figure 5: The process of selection of pair of vertices for Lemma 5.2.

The Möbius Ladder M_{4n+2} is a cubic graph of order $4n + 2$ with $6n + 3$ edges. The index function for Möbius Ladder M_{4n+2} becomes

$$N(x, y) = |6n + 3 - 12x - 8y|, \quad (5.1)$$

where $0 \leq x \leq \lfloor 4n/5 \rfloor$ is the number of non-adjacent pairs of vertices (c_i, d_i) with $c_i \in S$, $d_i \in T$ and $0 \leq y \leq n$ is the number of edges $u_i v_i$ such that $u_i \in S$, $v_i \in T$ and $A(u_i, v_j) = 0$ for all $i \neq j$. The friendly index sets of Möbius Ladders are known [12]. However, we wish to present it here to demonstrate how useful the index function is.

Theorem 5.3. $C(M_{4n+2}) = \{6n + 3\} \cup \{2i + 1 : 0 \leq i \leq 3n - 3\}$.

Proof. The proof is similar to the previous ones. If we arrange the values of the index function in a table, such as Table 6, we realize that the index function provides all the indices of M_{4n+2} except for three of them. Namely, $6n - 7$, $5n - 11$, and $6n - 13$. However, these numbers can easily be obtained by the following three colorings illustrated in Figures 6, 7 and 8. $\square$

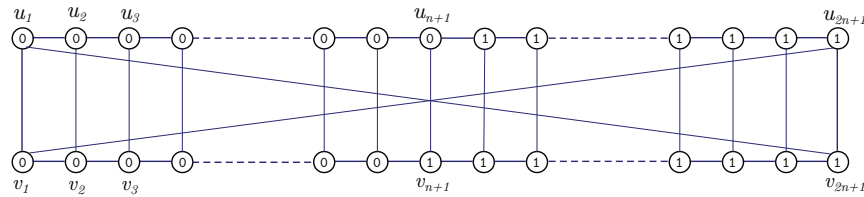


Figure 6: The coloring of M_{4n+2} that provides the index $6n - 7$.

For the Möbius Ladder M_{30} , the index function becomes $N(x, y) = |45 - 12x - 8y|$. Table 6 shows that all the indices are produced by this index function except for numbers 35, 31 and 27. One can independently produce these three numbers by the colorings demonstrated in Figures 6, 7 and 8.

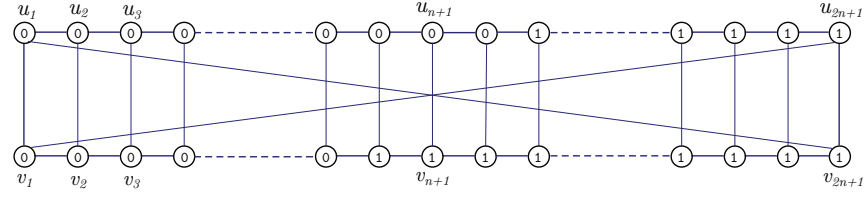


Figure 7: The coloring of M_{4n+2} that provides the index $6n - 11$.

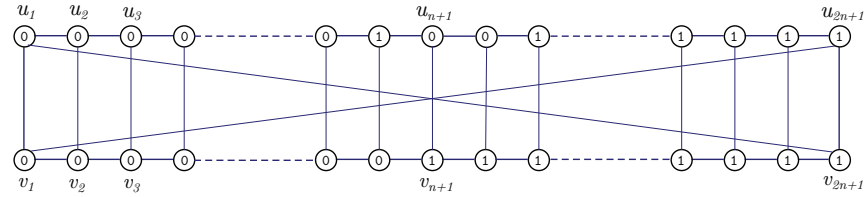


Figure 8: The coloring of M_{4n+2} that provides the index $6n - 15$.

$N(x, y)$	$y = 0$	1	2	3	4	5	6	7
$x = 0$	45	37	29	21	13	5	3	11
$x = 1$	33	25	17	9	1	7		
$x = 2$	21	13	5	3				
$x = 3$	9	1	7					
$x = 4$	3	11	19					
$x = 5$	15	23						

Table 6: The friendly indices of M_{30} generated by Equation (5.1).

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