

On Modular Colorings of Caterpillars

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ABSTRACT

A modular k -coloring, $k \geq 2$, of a graph G without isolated vertices is a coloring of the vertices of G with the elements in $\mathbb{Z}_k$ (where adjacent vertices may be colored the same) having the property that for every two adjacent vertices of G , the sums of the colors of their neighbors are different in $\mathbb{Z}_k$. The minimum k for which G has a modular k -coloring is the modular chromatic number $\text{mc}(G)$ of G . The modular chromatic number of a graph is at least as large as its chromatic number. It was known that if T is a nontrivial tree, then $\text{mc}(T) = 2$ or $\text{mc}(T) = 3$. A nontrivial tree T is of type one if $\text{mc}(T) = 2$ and is of type two if $\text{mc}(T) = 3$. It is shown that all nontrivial trees of diameter at most 6 are of type one. A caterpillar is a tree of order 3 or more, the removal of whose end-vertices produces a path. A characterization has been established for all caterpillars that are of type two.

Key Words: caterpillar, modular coloring, modular chromatic number.

AMS Subject Classification: 05C05, 05C15.

1 Introduction

For a vertex v of a graph G , let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). For a graph G without isolated vertices, let $c : V(G) \rightarrow \mathbb{Z}_k$ ($k \geq 2$) be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* $\sigma(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$, that is,

$$\sigma(v) = \sum_{u \in N(v)} c(u).$$

The coloring c is called a *modular sum k -coloring* or simply a *modular k -coloring* of G if $\sigma(x) \neq \sigma(y)$ in $\mathbb{Z}_k$ for all pairs x, y of adjacent vertices of G . A coloring c is a *modular coloring* if c is a modular k -coloring for some integer $k \geq 2$. The *modular chromatic number* $\text{mc}(G)$ of G is the minimum k for which G has a modular k -coloring. These concepts were introduced and studied in [2].

It is observed in [2] that $\text{mc}(G) \geq \chi(G)$ for every nontrivial graph G , where $\chi(G)$ is the chromatic number of a graph G . While each nontrivial tree has chromatic number 2, not every tree has modular chromatic number 2. It is shown in [2] that the tree T^* in Figure 1 has $\text{mc}(T^*) = 3$ and a modular 3-coloring of T^* is shown in Figure 1 (where the color of a vertex is placed within the vertex) together with the color sum $\sigma(v)$ for each vertex v of G (where the color sum of a vertex is placed next to the vertex). On the other hand, every nontrivial tree has modular chromatic number 2 or 3.

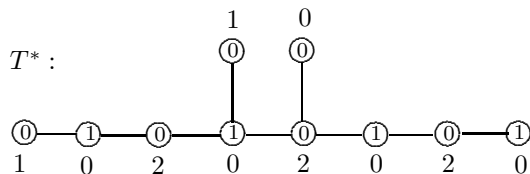


Figure 1: A tree with modular chromatic number 3

Theorem 1.1 [2] *If T is a nontrivial tree, then $\text{mc}(T) = 2$ or $\text{mc}(T) = 3$.*

A nontrivial tree T is of *type one* if $\text{mc}(T) = 2$ and is of *type two* if $\text{mc}(T) = 3$. In this paper, we investigate the problem of determining which trees are of type one and which trees are of type two. We refer to the book [1] for graph theory notation and terminology not described in this paper.

2 Trees with Small Diameter

For vertices u and v in a connected graph G , the *distance* $d(u, v)$ between u and v is the length of a shortest $u - v$ path in G . The *diameter* $\text{diam}(G)$ of G is the largest distance between two vertices of G . Thus the tree T^* in Figure 1 has $\text{diam}(T^*) = 7$. We have seen that T^* has modular chromatic number 3 and so is of type two. Next, we show that every nontrivial tree of diameter at most 6 is of type one.

Theorem 2.1 *If T is a nontrivial tree with $\text{diam}(T) \leq 6$, then $\text{mc}(T) = 2$.*

Proof. If $\text{diam}(T) \leq 4$, then let u^* be a central vertex of T . Then a 2-coloring c_1 of T such that $c_1(v) = 1$ if and only if $v = u^*$ is a modular coloring of T . For the case where $\text{diam}(T) \in \{5, 6\}$, let X be the set of end-vertices in T .

If $\text{diam}(T) = 5$, then let u_1^* and u_2^* be the central vertices of T . If either $\deg u_1^*$ or $\deg u_2^*$ is odd, say the former, then a 2-coloring c_2 of T such that $c_2(v) = 1$ if and only if $v \in N(u_1^*)$ is a modular coloring of T . Hence, suppose that both $\deg u_1^*$ and $\deg u_2^*$ are even. Let

$$X_i = N(u_i^*) \cap X \quad \text{and} \quad Y_i = N(u_i^*) - X_i \quad \text{for } i = 1, 2.$$

Observe that each of Y_1 and Y_2 contains at least two vertices. If $|Y_1|$ or $|Y_2|$ is odd, say the former, then let c_3 be a 2-coloring of T such that $c_3(v) = 1$ if and only if $v \in Y_1$. If $|Y_1|$ and $|Y_2|$ are both even, we consider the following two cases.

Case 1. At least one of X_1 and X_2 is nonempty. Suppose that $X_1 \neq \emptyset$ and let $x \in X_1$. Then a 2-coloring c_4 of T such that $c_4(v) = 1$ if and only if $v \in Y_1 \cup \{x\}$ is a modular coloring of T .

Case 2. Both X_1 and X_2 are empty. Let

$$Y_2' = Y_2 - \{u_1^*\} = \{w_1, w_2, \dots, w_a\},$$

where $a = \deg u_2^* - 1$. Furthermore, for each vertex $w_i \in Y_2'$ let x_i be an end-vertex adjacent to w_i . Then a 2-coloring c_5 of T given by $c_5(v) = 1$ if and only if $v \in [Y_1 - \{u_2^*\}] \cup \{x_1, x_2, \dots, x_a\}$ is a modular coloring of T .

Finally, if $\text{diam}(T) = 6$, then let u^* be the central vertex of T . If $\deg u^*$ is odd, then consider a 2-coloring c_6 of T such that $c_6(v) = 1$ if and only if $v \in N(u^*)$. If $\deg u^*$ is even, then let

$$P : u_1, u_2, u_3, u_4 = u^*, u_5, u_6, u_7$$

be a longest path in T . Consider the set

$$S = [N(u_5) - \{u^*\}] - X = \{w_1, w_2, \dots, w_a\}.$$

Note that S is nonempty since $u_6 \in S$. For each i with $1 \leq i \leq a$, let x_i be an end-vertex adjacent to w_i . Now consider a 2-coloring c_7 of T such that $c_7(v) = 1$ if and only if $v \in [N(u^*) - \{u_5\}] \cup \{x_1, x_2, \dots, x_a\}$ and observe that c_7 is a modular coloring of T . This completes the proof. ■

By Theorem 2.1, if a tree is of type two, then its diameter must be at least 7 and so its order must be at least 8.

3 Caterpillars

A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path called the *spine* of the caterpillar. Thus every path and star (of order at least 3) and every double star (a tree of diameter 3) is a caterpillar. For example, the tree T^* in Figure 1 is a caterpillar and we have seen that T^* is of type two. In this section, we determine which caterpillars are of type one and which caterpillars are of type two. By Theorem 2.1, every caterpillar of diameter at most 6 is of type one and so we consider caterpillars of diameter at least 7. It was shown in [2] that if G is a nontrivial connected bipartite graph containing no vertex of even degree, then any proper 2-coloring of G is a modular coloring of G . Thus we have the following.

Theorem 3.1 [2] *If G is a nontrivial connected bipartite graph containing no vertex of even degree, then $\text{mc}(G) = 2$.*

By Theorem 3.1, if T is a nontrivial tree containing no vertex of even degree, then T is of type one. It turns out that for caterpillars, containing no vertex of degree 2 guarantees that its modular chromatic number is 2 and so is of type one, which we show next.

Proposition 3.2 *If no vertex of a nontrivial caterpillar T has degree 2, then $\text{mc}(T) = 2$.*

Proof. By Proposition 2.1, we may assume that $d = \text{diam}(T) \geq 7$. Let $P : u_1, u_2, \dots, u_{d-1}$ be the spine of T and for each i ($1 \leq i \leq d-1$) let x_i be an end-vertex adjacent to u_i . Consider the coloring $c : V(T) \rightarrow \mathbb{Z}_2$ such that $c(v) = 1$ if and only if either (i) $v = u_i$ and i is odd or (ii) $v = x_j$ and j is even with $j \leq d-2$. Then c is a modular 2-coloring of T and so $\text{mc}(T) = 2$. ■

This result does not hold for general trees. For example, the tree T of Figure 2 contains no vertex having degree 2 but $\text{mc}(T) = 3$. If there is a modular 2-coloring c of T , then, by symmetry, we may assume that the color sums of the vertices are as shown in Figure 2. However then,

Since $\sigma(u_{\beta, n_\beta}) = 1$ and $c(u_{\beta, n_\beta-1}) = 1$, it follows that $c(u_{\beta+1, 1}) = 0$. However, this in turn implies that

$$\sigma(x_{\beta+1}) = c(u_{\beta+1, 1}) = 0 = \sigma(x_{\alpha+1}),$$

which is impossible since $d(x_{\alpha+1}, x_{\beta+1}) = D + n_\beta + 2$ is odd.

For the converse, suppose that for every two integers α and β with $1 \leq \alpha < \beta \leq s-1$, either

- (i) at least one of n_α and n_β is not congruent to 2 modulo 4 or
- (ii) $n_\alpha, n_\beta \equiv 2 \pmod{4}$ and $D = d(u_{\alpha+1, 1}, u_{\beta, 1})$ is even.

Let

$$N = \{n_1, n_2, \dots, n_{s-1}\} \text{ and } N^* = \{n_i \in N : n_i \equiv 2 \pmod{4}\}.$$

If $N^* = \emptyset$, then let c_0 be any proper 2-coloring of T . If $N^* \neq \emptyset$, then let

$$N^* = \{n_{i_1}, n_{i_2}, \dots, n_{i_t}\},$$

where $1 \leq i_1 < i_2 < \dots < i_t \leq s-1$. Consider a proper 2-coloring c_0 of T such that $c_0(u_{i_1, 1}) = 1$ and observe that $c_0(u_{i, 1}) = 1$ for each $i \in \{i_1, i_2, \dots, i_t\}$. For the tree T of Figure 3(a), such a coloring c_0 is shown in Figure 3(b). We then define a coloring $c : V(P) \rightarrow \mathbb{Z}_2$ as follows. For a fixed integer i ($1 \leq i \leq s$), consider a vertex $u_{i, j}$, where $1 \leq j \leq n_i$.

Case 1. $c_0(u_{i, 1}) = 0$. Then $n_i \not\equiv 2 \pmod{4}$. Let $c(u_{i, j}) = 1$ if and only if $j \equiv 1 \pmod{4}$. In particular, observe that $c(u_{i, 1}) = 1 \neq c_0(u_{i, 1})$.

Case 2. $c_0(u_{i, 1}) = 1$. If $n_i \equiv 1 \pmod{4}$, then let $c(u_{i, j}) = 1$ if and only if $j \equiv 2 \pmod{4}$. Otherwise let $c(u_{i, j}) = 1$ if and only if $j \equiv 0 \pmod{4}$. In each case, observe that $c(u_{i, 1}) = 0 \neq c_0(u_{i, 1})$.

The coloring c is shown in Figure 3(c) for the tree T of Figure 3(a), where the color assigned by c to each vertex of the spine of T is indicated inside the vertex. For every vertex $v \in V(P) - U$ where U is described in (1), its color sum $\sigma(v)$ is now determined and equal to $c_0(v)$ (which is indicated outside the vertex v in Figure 3(c)). Furthermore, for each end-vertex $x \in X_i$ where X_i is described in (2), observe that $\sigma(x) = c(u_{i, 1}) \neq c_0(u_{i, 1})$, implying that $\sigma(x) = c_0(x)$.

Finally, assign an appropriate color (0 or 1) to each of the end-vertices such that $\sigma(u) = c_0(u)$ for every $u \in U$. Figure 3(d) shows the color assignments by c to the end-vertices of the tree T of Figure 3(a), which are indicated inside the end-vertices and the color sum for each vertex $u \in U$ is indicated outside the vertex. Hence, the coloring we obtained is a modular 2-coloring of T . ■

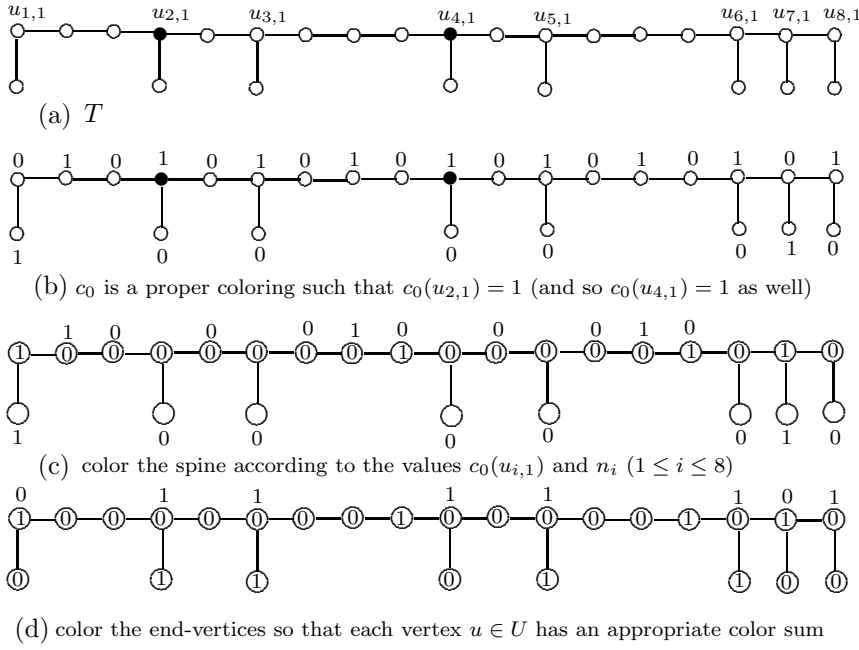


Figure 3: The modular 2-coloring of a tree T described in the proof of Theorem 3.3

By Theorem 3.3, the tree T^* in Figure 1 is a caterpillar of the smallest order (namely order 10) that is of type two. In fact, more can be said.

Proposition 3.4 *The tree T^* in Figure 1 is the tree of the smallest order that is of type two.*

Proof. Suppose that T is a tree of order at most 10 whose diameter is at least 7. If $\text{diam}(T) \geq 8$, then T is a caterpillar and $\text{mc}(T) = 2$ by Theorem 3.3. Hence, suppose that $\text{diam}(T) = 7$ and $\text{mc}(T) = 3$. If T is a caterpillar, then $T = T^*$ by Theorem 3.3. Otherwise, T is either T_1 or T_2 shown in Figure 4. Since each tree T_i ($i = 1, 2$) has a modular 2-coloring (shown in Figure 4), it follows that $\text{mc}(T_i) = 2$. So each of T_1 and T_2 are of type one. ■

The following is another consequence of Theorem 3.3.

Corollary 3.5 *There exists a tree T whose diameter is d and $\text{mc}(T) = 3$ if and only if $d \geq 7$.*

We conclude with the following problem.

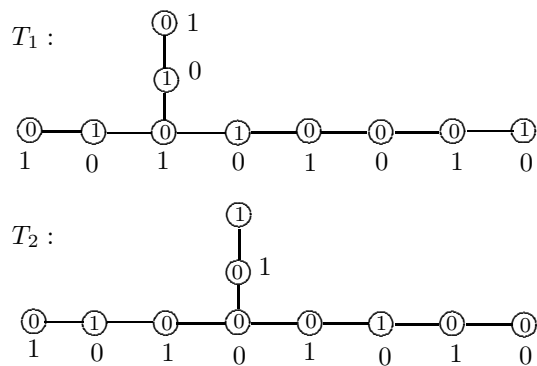


Figure 4: Two trees of order 10 that are of type one

Problem 3.6 *Is there a characterization of trees that are of type one (or a characterization of trees that are of type two)?*

References

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