A Solution to the Checkerboard Problem

Futaba Okamoto

Mathematics Department,
University of Wisconsin – La Crosse, La Crosse, WI 54601

Ebrahim Salehi

Department of Mathematical Sciences,
University of Nevada Las Vegas, Las Vegas, NV 89154

Ping Zhang

Department of Mathematics,
Western Michigan University, Kalamazoo, MI 49008

Abstract

An $m \times n$ checkerboard has m rows and n columns and its squares are alternately colored black and red. Two squares are said to be neighboring if they belong to the same row or to the same column and there is no square between them. A combinatorial problem called the Checkerboard Conjecture states that it is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity. In this work, we show that the Checkerboard Conjecture is true for all m and n.

AMS subject classification: 05B99.
Keywords: Checkerboard conjecture, solution.

1. Introduction

Suppose that the squares of an $m \times n$ checkerboard (m rows and n columns), where $mn \geq 2$, are alternately colored black and red. Figure 1(a) shows a 3×8 checkerboard where a shaded square represents a black square. Two squares are said to be *neighboring* if they belong to the same row or to the same column and there is no square between

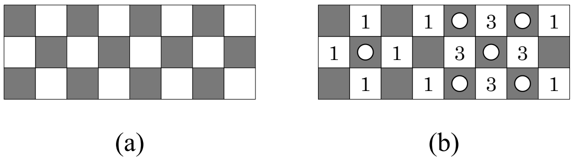


Figure 1: A 3 × 8 checkerboard and a coin placement on the checkerboard

them. Thus every two neighboring squares are of different colors. A combinatorial problem was introduced in [2] and the following conjecture was stated.

The Checkerboard Conjecture. It is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.

Figure 1(b) shows a placement of 6 coins on a 3 × 8 checkerboard such that the number of coins on neighboring squares of every black square is even and the number of coins on neighboring squares of every red square is odd. Thus for every two squares of different colors, the numbers of coins on neighboring squares are of opposite parity. Consequently, the Checkerboard Conjecture is true for a 3 × 8 checkerboard. Observe that all 6 coins on the 3×8 checkerboard of Figure 1(b) are placed only on black squares. Thus the number of coins on neighboring squares of every black square is 0, while the number of coins on neighboring squares of every red square is either 1 or 3 as indicated in Figure 1(b). Indeed, for any $m \times n$ checkerboard for which the Checkerboard Conjecture is true, there is always a solution in which all coins are placed only on squares of the same color. In this work, we show that the Checkerboard Conjecture is true for a checkerboard of any size.

The Checkerboard Theorem. For every pair m, n of positive integers, it is possible to place coins on some of the squares of an $m \times n$ checkerboard (at most one coin per square) such that for every two squares of the same color the numbers of coins on neighboring squares are of the same parity, while for every two squares of different colors the numbers of coins on neighboring squares are of opposite parity.

2. A Proof of the Checkerboard Theorem

In order to verify the Checkerboard Theorem, we first introduce some additional definitions and notation and present some preliminary results in Section 2.1. We then study certain extensions of coin placements in Section 2.2 and present a proof of the Checkerboard Theorem in Section 2.3.

2.1. Definitions, Notation, and Preliminary Results

We always assume that m and n are positive integers with $m \leq n$ and $n \geq 2$ unless otherwise noted. For an $m \times n$ checkerboard C, let $S = B \cup R$ be the set of mn squares in C, where B and R are the sets of black and red squares, respectively, and let $s_{i,j} \in S$ be the square in the ith row and jth column for $1 \leq i \leq m$ and $1 \leq j \leq n$. We may also assume that $B = \{s_{i,j} \in S : i + j \equiv 0 \pmod 2\}$.

We express a coin placement for C using a coin placement function $f : S \to \{0, 1\}$ defined by $f(s) = 1$ if and only if there is a coin placed on the square s. The corresponding *neighbor sum* of a square s, denoted by $\sigma_f(s)$ (or simply $\sigma(s)$), is the number of coins placed on the neighboring square of s. For simplicity, we further assume that $\sigma(s)$ is expressed as one of 0 and 1 modulo 2.

We say that a coin placement f for a checkerboard C is a *solution* if either $\sigma(s) = 1$ if and only if $s \in R$ or $\sigma(s) = 1$ if and only if $s \in B$. Therefore, the Checkerboard Conjecture is true for an $m \times n$ checkerboard C if C has a solution.

Let $S = S_1 \cup S_2 \cup \cdots \cup S_n$, where $S_j = \{s_{i,j} : 1 \leq i \leq m\}$ for $1 \leq j \leq n$. Hence, S_j is the set of the m squares in the jth column. Furthermore, let $S'_j = S_1 \cup S_2 \cup \cdots \cup S_j$ for $1 \leq j \leq n$. (Hence $S'_1 = S_1$ and $S'_n = S$.) Let $f_1 : S'_1 \to \{0, 1\}$ be an arbitrary coin placement for the squares in S'_1 such that either $f_1(s) = 0$ for every $s \in S'_1 \cap R$ or $f_1(s) = 0$ for every $s \in S'_1 \cap B$, say the former. Observe then that there exists a unique coin placement $f_2 : S'_2 \to \{0, 1\}$ such that

(i) $f_2(s) = 0$ for every $s \in S'_2 \cap R$,

(ii) f_2 restricted to S'_1 equals f_1, and

(iii) $\sigma(s) = 1$ for every $s \in S'_1 \cap R$.

After finding such a coin placement f_2, observe further that there exists a unique coin placement $f_3 : S'_3 \to \{0, 1\}$ such that

(i) $f_3(s) = 0$ for every $s \in S'_3 \cap R$,

(ii) f_3 restricted to S'_2 equals f_2, and

(iii) $\sigma(s) = 1$ for every $s \in S'_2 \cap R$.

In general, for every integer j $(1 \leq j \leq n - 1)$ suppose that $f_j : S'_j \to \{0, 1\}$ is a coin placement such that $f_j(s) = 0$ for every $s \in S'_j \cap R$ and $\sigma(s) = 1$ for every $s \in S'_{j-1} \cap R$ (if $j \geq 2$). Then there exists a unique coin placement $f_{j+1} : S'_{j+1} \to \{0, 1\}$ such that

(i) $f_{j+1}(s) = 0$ for every $s \in S'_{j+1} \cap R$,

(ii) f_{j+1} restricted to S'_j equals f_j, and

(iii) $\sigma(s) = 1$ for every $s \in S'_j \cap R$.

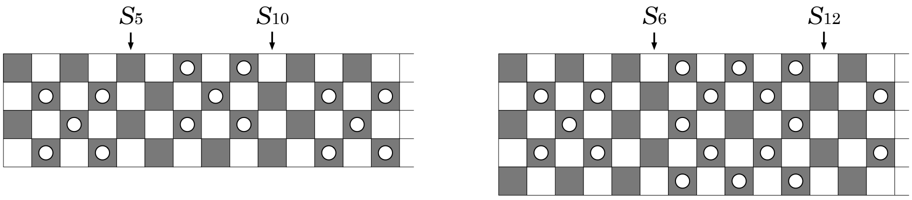


Figure 2: Trivial extensions for $m = 4, 5$

This yields the following lemma.

Lemma 2.1. Consider an $m \times n$ checkerboard. For each coin placement $f_1 : S_1 \to \{0, 1\}$ such that $f_1(s) = 0$ for every $s \in S_1 \cap R$, there exists a unique coin placement $F : S \to \{0, 1\}$ such that

(i) $F(s) = 0$ for every $s \in S \cap R$,

(ii) F restricted to S_1 equals f_1, and

(iii) $\sigma(s) = 1$ for every $s \in S'_{n-1} \cap R$.

(Similarly, for each coin placement $f_1 : S_1 \to \{0, 1\}$ such that $f_1(s) = 0$ for every $s \in S_1 \cap B$, there exists a unique coin placement $F : S \to \{0, 1\}$ such that

(i) $F(s) = 0$ for every $s \in S \cap B$,

(ii) F restricted to S_1 equals f_1, and

(iii) $\sigma(s) = 1$ for every $s \in S'_{n-1} \cap B$.)

For the coin placements f_1 and F for a checkerboard C described in Lemma 2.1, we say that F is the *extension* of f_1. We also say that F is an extension for C.

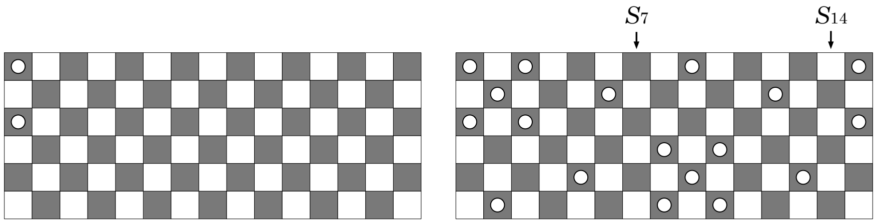
2.2. Properties of Extensions of Coin Placements

Consider an $m \times n$ checkerboard, where n is sufficiently large. Let $f_1 : S_1 \to \{0, 1\}$ be an arbitrary coin placement such that $f_1(s) = 0$ for every $s \in S_1 \cap R$ and obtain the unique extension F of f_1. We will next show the following:

$$\text{If } j \equiv 0 \pmod{m+1}\text{, then } F(s) = 0 \text{ for } \textit{every} \text{ square } s \in S_j. \tag{2.1}$$

Definition 2.2. For the trivial coin placement $f_1^* : S_1 \to \{0, 1\}$ such that $f^*(s) = 0$ for every $s \in S_1$, its unique extension is called the trivial extension and denoted by F^*.

See Figure 2 shows the trivial extensions for $m = 4, 5$ and note that both extensions satisfy (2.1). In fact, the trivial extension satisfies (2.1) for every m, which we state without a proof as follows.



(a) (b)

Figure 3: A coin placement f_1 and its extension F for a 6×15 checkerboard.

Observation 2.3. If F^* is the trivial extension for an $m \times n$ checkerboard, then $F^*(s) = 0$ for every $s \in S_j$, where $j \equiv 0 \pmod{m+1}$.

Before proving (2.1) for an arbitrary extension, let us look at another example. Consider a 6×15 checkerboard with a coin placement $f_1 : S_1 \to \{0, 1\}$ placing two coins on black squares as shown in Figure 3(a). Figure 3(b) shows the extension F of f_1. Observe for every square s except those in the last column that $\sigma(s) = 1$ if and only if $s \in R$. Furthermore, $F(s) = 0$ for every $s \in S_7 \cup S_{14}$.

Proposition 2.4. Let C be an $m \times n$ checkerboard with $m \leq n$. For an arbitrary coin placement $f_1 : S_1 \to \{0, 1\}$ with $f_1(s) = 0$ for every $s \in S_1 \cap R$, let $F : S \to \{0, 1\}$ be its extension. Then $F(s) = 0$ for every $s \in S_j$ where $j \equiv 0 \pmod{m+1}$.

Proof. Let $f_1 : S_1 \to \{0, 1\}$ be given by

$$f_1(s_{i,1}) = \begin{cases} a_{\lceil i/2 \rceil} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even} \end{cases}$$

for $1 \leq i \leq m$. Furthermore, for each integer j with $0 \leq j \leq \lceil m/2 \rceil$ let

$$A_j = \begin{cases} \displaystyle\sum_{i=1}^{j} a_i & \text{if } 1 \leq j \leq \lceil m/2 \rceil \\ 0 & \text{if } j = 0. \end{cases}$$

We consider two cases according to the parity of m.

Case 1. m is even. Define a coin placement $f : S \to \{0, 1\}$ such that $f(s) = 0$ for every $s \in R$ as follows. Suppose that i $(1 \leq i \leq m)$ and j $(1 \leq j \leq n)$ are positive integers such that $i + j$ is even. (Hence $s_{i,j} \in B$.) Then

$$f(s_{i,j}) = \begin{cases} A_{\frac{i+j}{2}} + A_{\frac{i-j}{2}} + F^*(s_{i,j}) & \text{if } j \le i \text{ and } i + j \le m \\ A_{(m+1)-\frac{i+j}{2}} + A_{\frac{i-j}{2}} + F^*(s_{i,j}) & \text{if } j \le i \text{ and } i + j \ge m + 2 \\ f(s_{j,i}) & \text{if } i + 2 \le j \le m \\ 0 & \text{if } j = m + 1 \\ f(s_{(m+1)-i,j-(m+1)}) & \text{if } m + 2 \le j \le 2m + 2 \\ f(s_{i,j-(2m+2)}) & \text{if } j \ge 2m + 3, \end{cases}$$

where F^* is the trivial extension defined in Definition 2.2. Note that

$$f(s_{i,2m+2}) = f(s_{(m+1)-i,m+1}) = 0$$

and so $f(s_{i,j}) = 0$ whenever $j \equiv 0 \pmod{m+1}$. Also

$$f(s_{i,1}) = A_{\frac{i+1}{2}} + A_{\frac{i-1}{2}} + F^*(s_{i,1}) = a_{\frac{i+1}{2}} + 0 = c_1(s_{i,1})$$

for $i = 1, 3, \ldots, m - 1$ and so f restricted to S_1 equals f_1.

We now show that f is the extension of f_1. To do this, we need only verify that $\sigma(s_{i,j}) = 1$ for every $s_{i,j} \in S'_{n-1} \cap R$. That is, we show that $\sigma(s_{i,j}) = 1$ for integers i and j with $1 \le i \le m$ and $1 \le j \le n - 1$ such that $i + j$ is odd. By symmetry, we may further suppose that either (i) $1 \le j < i \le m$ or (ii) $j = m + 1$.

Subcase 1.1. $1 \le j < i \le m$**.** First suppose that $j = 1$. If $2 \le i \le m - 2$, then

$$\begin{aligned} \sigma(s_{i,1}) &= f(s_{i,2}) + f(s_{i-1,1}) + f(s_{i+1,1}) \\ &= \left[A_{\frac{i+2}{2}} + A_{\frac{i-2}{2}} + F^*(s_{i,2})\right] + \left[A_{\frac{i}{2}} + A_{\frac{i-2}{2}} + F^*(s_{i-1,1})\right] \\ &\quad + \left[A_{\frac{i+2}{2}} + A_{\frac{i}{2}} + F^*(s_{i+1,1})\right] \\ &= F^*(s_{i,2}) + F^*(s_{i-1,1}) + F^*(s_{i+1,1}) = 1 + 0 + 0 = 1, \end{aligned}$$

while

$$\begin{aligned} \sigma(s_{m,1}) &= f(s_{m,2}) + f(s_{m-1,1}) \\ &= \left[A_{(m+1)-\frac{m+2}{2}} + A_{\frac{m-2}{2}} + F^*(s_{m,2})\right] + \left[A_{\frac{m}{2}} + A_{\frac{m-2}{2}} + F^*(s_{m-1,1})\right] \\ &= F^*(s_{m,2}) + F^*(s_{m-1,1}) = 1 + 0 = 1. \end{aligned}$$

Next suppose that $i = m$ and $3 \le j \le m - 1$. Then

$$\begin{aligned} \sigma(s_{m,j}) &= f(s_{m,j-1}) + f(s_{m,j+1}) + f(s_{m-1,j}) \\ &= \left[A_{(m+1)-\frac{m+j-1}{2}} + A_{\frac{m-j+1}{2}} + F^*(s_{m,j-1})\right] \\ &\quad + \left[A_{(m+1)-\frac{m+j+1}{2}} + A_{\frac{m-j-1}{2}} + F^*(s_{m,j+1})\right] \\ &\quad + \left[A_{(m+1)-\frac{m+j-1}{2}} + A_{\frac{m-j-1}{2}} + F^*(s_{m-1,j})\right] \\ &= F^*(s_{m,j-1}) + F^*(s_{m,j+1}) + F^*(s_{m-1,j}) = 1 + 1 + 1 = 1. \end{aligned}$$

Hence, suppose next that $2 \le j < i \le m-1$. If $i+j \le m-1$, then

$$\begin{aligned}\sigma(s_{i,j}) &= f(s_{i,j-1}) + f(s_{i,j+1}) + f(s_{i-1,j}) + f(s_{i+1,j}) \\ &= \left[A_{\frac{i+j-1}{2}} + A_{\frac{i-j+1}{2}} + F^*(s_{i,j-1})\right] + \left[A_{\frac{i+j+1}{2}} + A_{\frac{i-j-1}{2}} + F^*(s_{i,j+1})\right] \\ &\quad + \left[A_{\frac{i+j-1}{2}} + A_{\frac{i-j-1}{2}} + F^*(s_{i-1,j})\right] + \left[A_{\frac{i+j+1}{2}} + A_{\frac{i-j+1}{2}} + F^*(s_{i+1,j})\right] \\ &= F^*(s_{i,j-1}) + F^*(s_{i,j+1}) + F^*(s_{i-1,j}) + F^*(s_{i+1,j}) = 1.\end{aligned}$$

For $i+j = m+1$,

$$\begin{aligned}\sigma(s_{i,j}) &= f(s_{i,j-1}) + f(s_{i,j+1}) + f(s_{i-1,j}) + f(s_{i+1,j}) \\ &= \left[A_{\frac{m}{2}} + A_{\frac{i-j+1}{2}} + F^*(s_{i,j-1})\right] + \left[A_{(m+1)-\frac{m+2}{2}} + A_{\frac{i-j-1}{2}} + F^*(s_{i,j+1})\right] \\ &\quad + \left[A_{\frac{m}{2}} + A_{\frac{i-j-1}{2}} + F^*(s_{i-1,j})\right] + \left[A_{(m+1)-\frac{m+2}{2}} + A_{\frac{i-j+1}{2}} + F^*(s_{i+1,j})\right] \\ &= F^*(s_{i,j-1}) + F^*(s_{i,j+1}) + F^*(s_{i-1,j}) + F^*(s_{i+1,j}) = 1.\end{aligned}$$

Similarly, if $i+j \ge m+3$, then

$$\begin{aligned}\sigma(s_{i,j}) &= f(s_{i,j-1}) + f(s_{i,j+1}) + f(s_{i-1,j}) + f(s_{i+1,j}) \\ &= \left[A_{(m+1)-\frac{i+j-1}{2}} + A_{\frac{i-j+1}{2}} + F^*(s_{i,j-1})\right] \\ &\quad + \left[A_{(m+1)-\frac{i+j+1}{2}} + A_{\frac{i-j-1}{2}} + F^*(s_{i,j+1})\right] \\ &\quad + \left[A_{(m+1)-\frac{i+j-1}{2}} + A_{\frac{i-j-1}{2}} + F^*(s_{i-1,j})\right] \\ &\quad + \left[A_{(m+1)-\frac{i+j+1}{2}} + A_{\frac{i-j+1}{2}} + F^*(s_{i+1,j})\right] \\ &= F^*(s_{i,j-1}) + F^*(s_{i,j+1}) + F^*(s_{i-1,j}) + F^*(s_{i+1,j}) = 1.\end{aligned}$$

Subcase 1.2. $j = m+1$**.** Then i is even and $2 \le i \le m$. If $2 \le i \le m-2$, then

$$\begin{aligned}\sigma(s_{i,m+1}) &= f(s_{i,m}) + f(s_{i,m+2}) + f(s_{i-1,m+1}) + f(s_{i+1,m+1}) \\ &= f(s_{m,i}) + f(s_{m-i+1,1}) + 0 + 0 \\ &= \left[A_{(m+1)-\frac{m+i}{2}} + A_{\frac{m-i}{2}} + F^*(s_{m,i})\right] \\ &\quad + \left[A_{\frac{m-i+2}{2}} + A_{\frac{m-i}{2}} + F^*(s_{m-i+1,1})\right] \\ &= F^*(s_{m,i}) + F^*(s_{m-i+1,1}) = 1 + 0 = 1.\end{aligned}$$

Finally,

$$\begin{aligned}\sigma(s_{m,m+1}) &= f(s_{m,m}) + f(s_{m,m+2}) + f(s_{m-1,m+1}) = f(s_{m,m}) + f(s_{1,1}) + 0 \\ &= \left[A_{(m+1)-m} + A_0 + F^*(s_{m,m})\right] + \left[A_1 + A_0 + F^*(s_{1,1})\right] \\ &= F^*(s_{m,m}) + F^*(s_{1,1}) = 1 + 0 = 1.\end{aligned}$$

Therefore, $f = F$ by Lemma 2.1.

Case 2. m is odd. Define a coin placement $f : S \to \{0, 1\}$ such that $f(s) = 0$ for every $s \in R$ as follows. Suppose that i $(1 \le i \le m)$ and j $(1 \le j \le n)$ are positive integers such that $i + j$ is even. Then

$$f(s_{i,j}) = \begin{cases} A_{\frac{i+j}{2}} + A_{\frac{i-j}{2}} + F^*(s_{i,j}) & \text{if } j \le i \text{ and } i+j \le m+1 \\ A_{(m+1)-\frac{i+j}{2}} + A_{\frac{i-j}{2}} + F^*(s_{i,j}) & \text{if } j \le i \text{ and } i+j \ge m+3 \\ f(s_{j,i}) & \text{if } i+2 \le j \le m \\ 0 & \text{if } j \in \{m+1, 2m+2\} \\ A_{(m+1)-\frac{i+j}{2}} + A_{\frac{j-i}{2}} + F^*(s_{i,j}) & \text{if } m+2 \le j \le 2m+2-i \\ & \quad \text{and } j-i \le m+1 \\ A_{(m+1)-\frac{i+j}{2}} + A_{(m+1)-\frac{j-i}{2}} + F^*(s_{i,j}) & \text{if } m+2 \le j \le 2m+2-i \\ & \quad \text{and } j-i \ge m+3 \\ f(s_{(2m+2)-j,(2m+2)-i}) & \text{if } 2m+4-i \le j \le 2m+1 \\ f(s_{i,j-(2m+2)}) & \text{if } j \ge 2m+3. \end{cases}$$

Note that $f(s_{i,j}) = 0$ whenever $j \equiv 0 \pmod{m+1}$. Also

$$f(s_{i,1}) = A_{\frac{i+1}{2}} + A_{\frac{i-1}{2}} + F^*(s_{i,1}) = a_{\frac{i+1}{2}} + 0 = f_1(s_{i,1})$$

for $i = 1, 3, \ldots, m$ and so f restricted to S_1 equals f_1. One can also verify that $f = F$ by an argument similar to the one used in Case 1. ■

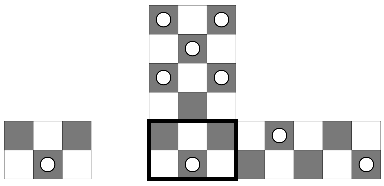
Proposition 2.4 yields the following useful result.

Corollary 2.5. Let F be an extension of an $m \times n$ checkerboard such that $F(s) = 0$ for every $s \in R$. If ℓ is a positive integer such that $\{s_{1,\ell}, s_{m,\ell}\} \not\subseteq B$ and $F(s) = 0$ for every $s \in S_\ell$, then $F(s) = 0$ for every $s \in S_j$, where $j \equiv \ell \pmod{m+1}$.

2.3. New Solutions from Old and the Main Result

In this section we study how one can obtain a solution for a checkerboard of certain size from a solution for another checkerboard of different size, which will lead to a proof of the Checkerboard Theorem.

Figure 4(a) shows a solution for a 2×3 checkerboard such that $\sigma(s) = 1$ if and only if $s \in R$. Extending this coin placement, we are able to obtain a solution for a 2×8 checkerboard as well as a solution for a 3×6 checkerboard, both of which has the property that $\sigma(s) = 1$ if and only if $s \in R$, as shown in Figure 4(b). Therefore, the solution for the 2×3 checkerboard in Figure 4(a) can be *extended* to solutions for 2×8 and 3×6 checkerboards. On the other hand, we may also say that the solution for the 2×8 checkerboard shown in Figure 4(b) can be *reduced* to a solution for a 2×3 checkerboard.



(a) (b)

Figure 4: Examples of extension and reduction of solutions

Recall that if f is a solution for an $m \times n$ checkerboard, then $\sigma_f(s) = 1$ if and only if $s \in R$ (or for every $s \in B$); while if F is an extension for an $m \times n$ checkerboard, then $\sigma_F(s) = 1$ for every $s \in R$ (for every $s \in B$) except possibly for those in $S_n \cap R$ (in $S_n \cap B$). The following observation describes how an extension of an $m \times n$ checkerboard induces a solution for an $m \times \ell$ checkerboard for some special values of $\ell < n$.

Observation 2.6. If F is an extension for an $m \times n$ checkerboard satisfying that $F(s) = 0$ for every $s \in S_{\ell+1}$, then an $m \times \ell$ checkerboard has a solution.

As a consequence of Proposition 2.4 and Observation 2.6, we obtain a result for $m \times m$ checkerboards.

Corollary 2.7. For every integer $m \geq 2$, an $m \times m$ checkerboard C has a solution such that $\sigma(s) = 1$ if and only if $s \in R$.

The following result will be useful to us.

Proposition 2.8. If $n \equiv \ell \pmod{m+1}$, then an $m \times \ell$ checkerboard has a solution with $\sigma(s) = 1$ for every $s \in R$ if and only if an $m \times n$ checkerboard has a solution with $\sigma(s) = 1$ for every $s \in R$.

Proof. It suffices to show that an $m \times \ell$ checkerboard has a solution with $\sigma(s) = 1$ for every $s \in R$ if and only if an $m \times (\ell + m + 1)$ checkerboard has a solution with $\sigma(s) = 1$ for every $s \in R$.

First consider an $m \times \ell$ checkerboard with a solution f such that $\sigma(s) = 1$ if and only if $s \in R$. Let F_R and F_L be the coin placement obtained by extending f to the right and to the left, respectively. If either m is even or ℓ is odd, then $\{s_{1,\ell+1}, s_{m,\ell+1}\} \not\subseteq B$ and $F_R(s) = 0$ for every $s \in S_{\ell+1}$, implying that $F_R(s) = 0$ for every $s \in S_{\ell+m+2}$ by Corollary 2.5. Then the result follows by Observation 2.6.

For the converse, suppose that C is an $m \times (\ell + m + 1)$ checkerboard having a solution f such that $\sigma(s) = 1$ if and only if $s \in R$. Then $f(s) = 0$ for every $s \in S_{m+1}$ by Proposition 2.4. Therefore, f restricted to the last ℓ columns of C is a solution for an $m \times \ell$ checkerboard with the desired property. ■

Before finally proving the main theorem, we show that the Checkerboard Conjecture holds for checkerboards of special size.

Lemma 2.9. If $n \equiv 1, m, m \pm 1 \pmod{m+1}$, then an $m \times n$ checkerboard has a solution such that $\sigma(s) = 1$ for every $s \in R$.

Proof. By Corollary 2.7 and Proposition 2.8 it suffices to show that an $m \times k$ checkerboard has a solution such that $\sigma(s) = 1$ if and only if $s \in R$ for $k \in \{1, m \pm 1\}$. For a $1 \times m$ checkerboard, the coin placement f defined by

$$f(s_{1,j}) = 1 \text{ if and only if } \begin{cases} j \equiv 3 & \text{if } m \equiv 0 \pmod 4 \\ j \equiv 1 & \text{otherwise} \end{cases}$$

is a solution. For an $m \times (m+1)$ checkerboard, the trivial extension F^* is a solution if m is even; otherwise, the extension of the coin placement $f_1 : S_1 \to \{0, 1\}$ defined by $f_1(s) = 0$ if and only if $s \in S_1 \cap R$ is a solution. This also implies that an $m \times (m-1)$ checkerboard has the desired property. ■

Theorem 2.10. [The Checkerboard Theorem] An $m \times n$ checkerboard has a solution for every pair m, n of positive integers with $mn \geq 2$.

Proof. We in fact show that every checkerboard has a solution such that $\sigma(s) = 1$ for every $s \in R$. Assume, to the contrary, that there exists an $m \times n$ checkerboard C $(m \leq n)$ with no such solution. Therefore, $n \geq m + 3$ by Lemma 2.9. We may assume that C is a smallest such checkerboard, that is, every checkerboard having less than mn squares has a solution with $\sigma(s) = 1$ if and only if $s \in R$. In particular, an $m \times (n - m - 1)$ checkerboard has the desired property. However, Proposition 2.8 implies that so does C, which is a contradiction. ■

To actually find a solution for a given $m \times n$ checkerboard C, let $m = \ell_1$, $n = \ell_0$, where then $\ell_1 \leq \ell_0$, and $C = C_0$. Let ℓ_2 be the integer with $1 \leq \ell_2 \leq \ell_1 + 1$ and $\ell_0 \equiv \ell_2 \pmod{\ell_1 + 1}$. If $\ell_2 \in \{1, \ell_1, \ell_1 \pm 1\}$, then construct a solution for C_0 using Lemma 2.9. Otherwise, we consider an $\ell_2 \times \ell_1$ checkerboard C_1. Let ℓ_3 be the integer with $1 \leq \ell_3 \leq \ell_2 + 1$ and $\ell + 1 \equiv \ell_3 \pmod{\ell_2 + 1}$. If $\ell_3 \in \{1, \ell_2, \ell_2 \pm 1\}$, then construct a solution for C_1 using Lemma 2.9. Otherwise, consider an $\ell_3 \times \ell_2$ checkerboard C_2, and so on. We continue this process until we obtain an $\ell_{k+1} \times \ell_k$ checkerboard C_k $(k \geq 0)$ for which there exists a solution such that $\sigma(s) = 1$ if and only if $s \in R$. Then by extending C_k upward and rotating by 90°, we obtain the $\ell_k \times \ell_{k-1}$ checkerboard C_{k-1} with a solution such that $\sigma(s) = 1$ if and only if $s \in R$. In general, suppose that we have a solution for the $\ell_{i+1} \times \ell_i$ checkerboard C_i such that $\sigma(s) = 1$ if and only if $s \in R$ for some i $(1 \leq i \leq k)$. Reposition C_i, if necessary, so that $\{s_{1,1}, s_{1,\ell_i}\} \notin R$. Then extending C_i upward, we obtain a solution for an $\ell_{i-1} \times \ell_i$ checkerboard, which can be rotated to result in a solution for the $\ell_i \times \ell_{i-1}$ checkerboard C_{i-1} with the desired property.

We illustrate this process described above with the following two examples.

Example 2.11. Consider an 8×13 checkerboard. Since $13 \equiv 4 \pmod{8+1}$ $(4 \notin \{1, 8, 8 \pm 1\})$ and $8 \equiv 3 \pmod{4+1}$ $(3 \in \{1, 4, 4 \pm 1\})$, first obtain a solution for a 3×4 checkerboard C_2 as described in Lemma 2.9. Extending this solution results in

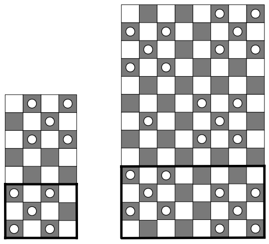


Figure 5: Constructing solutions for 4×8 and 8×13 checkerboards

a solution for a 8×4 checkerboard C_1, which can be extended again and we obtain a solution for an 8×13 checkerboard C_0 with a solution. (See Figure 5.)

Example 2.12. Consider an 11×18 checkerboard. Since $18 \equiv 6 \pmod{11+1}$ $(6 \notin \{1, 11, 11 \pm 1\})$, $11 \equiv 4 \pmod{6+1}$ $(4 \notin \{1, 6, 6 \pm 1\})$, and $6 \equiv 1 \pmod{4+1}$ $(1 \in \{1, 4, 4 \pm 1\})$, after obtaining a solution for a 1×4 checkerboard, one can recursively construct solutions for 4×6, 6×11, and 11×18 checkerboards, as shown in Figure 6.

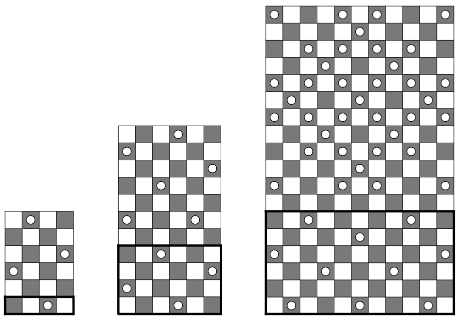


Figure 6: Constructing solutions for 4×6, 6×11, and 11×18 checkerboards

We conclude this paper with related open questions.

Problem 2.13. For a given checkerboard, how many solutions (up to symmetry) are there? Which checkerboard have unique solutions?

Problem 2.14. For a given checkerboard, what is the minimum number of coins necessary to construct a solution? Also, what is the maximum number of coins that can be used in a solution?

Acknowledgments

We are grateful to Professor Gary Chartrand for suggesting the problem to us and kindly providing useful information on this topic.

References

[1] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs: Fifth Edition*, Chapman & Hall/CRC, Boca Raton, FL, 2010.

[2] F. Okamoto, E. Salehi, and P. Zhang, A checkerboard problem and modular colorings of graphs. *Bull. Inst. Combin. Appl.*, **58**, pp. 29–47, 2010.