

PC-Labeling of a Graph and its PC-Set

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Abstract

A binary vertex labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of a graph G is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. This friendly labeling induces an edge labeling $f^* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(uv) = f(u)f(v)$ for all $uv \in E(G)$. Let $e_f(i) = \{uv \in E(G) : f^*(uv) = i\}$ be the number of edges of G that are labeled i . Product-cordial index of the labeling f is the number $pc(f) = |e_f(0) - e_f(1)|$. The product-cordial set of the graph G , denoted by $PC(G)$, is defined by

$$PC(G) = \{pc(f) : f \text{ is a friendly labeling of } G\}.$$

In this paper we will determine the product-cordial sets of certain classes of graphs.

Key Words: friendly labeling, product-cordial index, product-cordial set.

AMS Subject Classification: 05C78

1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [6]. Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}_2$ be a binary vertex coloring (labeling) of G . For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices colored 0 is almost the same as the number of vertices colored 1.

Any friendly coloring $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(xy) = f(x)f(y) \forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f^{*-1}(i)|$ be the number of edges of G that are labeled i . The number $pc(f) = |e_f(1) - e_f(0)|$ is called the *product-cordial index* (or *pc-index*) of f . The *product-cordial set* (or *pc-set*) of the graph G , denoted by $PC(G)$, is defined by

$$PC(G) = \{pc(f) : f \text{ is a friendly vertex coloring of } G\}.$$

To illustrate the above concepts, consider the graph G of Figure 1, which has six vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that three vertices be labeled 0 and the other three 1.

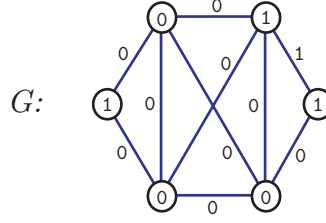


Figure 1: An example of product-cordial labeling of G .

Figure 1 also shows the associated edge labeling of G , where one edge has label 1 while the other 9 edges have labels 0. Therefore, the product-cordial index (or pc-index) of this labeling is $9 - 1 = 8$. It is easy to see that $PC(G) = \{4, 6, 8\}$. The friendly colorings of G that provide the other two pc-indices are presented in Figure 2.

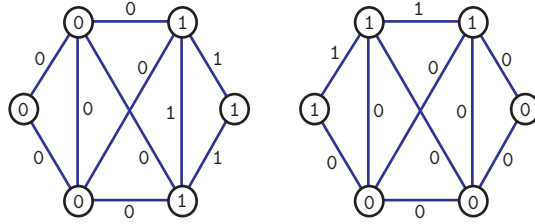


Figure 2: Two friendly labelings of G with pc-indices 4 and 6.

For a graph $G = (p, q)$ of size q , and a friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of G , we have

$$pc(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)|. \quad (1.1)$$

Therefore, to find the pc-index of f it is enough to find $e_f(1)$. Moreover, to determine the pc-set of G it is enough to compute $e_f(1)$ for different friendly labelings. Another immediate consequence of (1.1) is the following useful fact:

Observation 1.1. For a graph G of size q , $PC(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.

Definition 1.2. A graph G of size q is said to be fully product-cordial (or fully pc) if

$$PC(G) = \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}.$$

For example, the graph G of Figure 1 is not fully pc. However, P_n , the path of order n , is fully pc. In case of P_n , it is easy to observe that $e_f(1) = 0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor$, which proves that

Theorem 1.3. For any $n \geq 2$, $PC(P_n) = \{n - 1 - 2k : 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\}$.

The different friendly labelings of P_7 that provide its pc-set are illustrated in Figure 3.

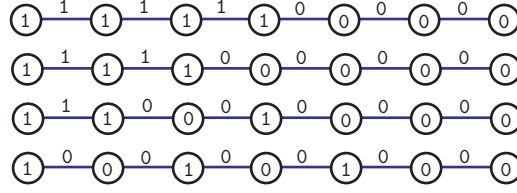


Figure 3: $PC(P_7) = \{0, 2, 4, 6\}$.

In 1978, I. Cahit [2, 3, 4] introduced the concept of cordial labeling as a weakened version of the less tractable graceful and harmonious labeling. Given a cordial labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of a graph G , Cahit introduced an edge labeling $f_+ : E(G) \rightarrow \mathbb{Z}_2$ by $f_+(uv) = |f(u) - f(v)|$ and defined the cordial index $c(f)$ of f to be $|f_+^{-1}(0) - f_+^{-1}(1)|$. A graph is called cordial if it admits a cordial labeling with cordial index 0 or 1. Cahit, among other facts, proved that

1. every tree is cordial;
2. The complete graph K_n is cordial if and only if $n \leq 3$;
3. The complete bipartite graph $K(m, n)$ is cordial ($m, n \in \mathbb{N}$);
4. The wheel W_n is cordial if and only if $n \not\equiv 3 \pmod{4}$;
5. In an Eulerian graph $G = (p, q)$ if $p \equiv 0 \pmod{4}$, then it is not cordial.

M. Hovay [9], later generalized the concept of cordial graphs and introduced A -cordial labelings. A graph G is said to be A -cordial if it admits a labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 5, 8, 10, 11, 14, 18].

Product cordial labeling of a graph was introduced by Sundaram, Ponraj and Somasundaran [21]. They call a graph G product-cordial if it admits a friendly labeling whose product-cordial index is at most 1. Then Sundaram, Ponraj and Somasundaran [21, 22, 23] investigate whether certain graphs such as trees, cycles, complete graphs, wheels, etc. are product-cordial. In this paper the product-cordial sets of certain classes of graphs are determined. Naturally, if $0 \in PC(G)$ or $1 \in PC(G)$, then G would be product-cordial.

2 PC Sets of Complete Graphs

Theorem 2.1. For any $n \geq 3$, $PC(K_n) = \{\lfloor n/2 \rfloor^2, \lfloor n/2 \rfloor^2 + \lfloor n/2 \rfloor(1 - (-1)^n)\}$.

Proof. Let $f : V(K_n) \rightarrow \mathbb{Z}_2$ be an arbitrary friendly labeling of K_n . We consider the following two cases:

(a) Let $n = 2\ell$ be even. Then ℓ vertices of K_n are labeled 1. Therefore, the induced edge labeling marks exactly $\ell(\ell - 1)/2$ edges of K_n by 1. That is, $e_f(1) = \ell(\ell - 1)/2$. Hence $pc(f) = \ell(2\ell - 1) - \ell(\ell - 1) = \ell^2 = \lfloor n/2 \rfloor^2$.

(b) Let $n = 2\ell + 1$ be odd. Then either $\ell + 1$ or ℓ vertices of K_n are labeled 1. Therefore, $e_f(1) = \ell(\ell + 1)/2$ or $\ell(\ell - 1)/2$. Hence $pc(f) = \ell^2 = \lfloor n/2 \rfloor^2$ or $\ell^2 + 2\ell = \lfloor n/2 \rfloor^2 + \lfloor n/2 \rfloor(1 - (-1)^n)$. $\square$

Theorem 2.2. For any $n \geq 4$, $PC(K_n - e) = \{\lfloor n/2 \rfloor^2 \pm 1, \lfloor n/2 \rfloor^2 \pm 1 + (1 - (-1)^n)\lfloor n/2 \rfloor\}$.

Proof. It is easy to verify that $PC(K_4 - e) = \{3, 5\}$. The two friendly coloring that provide these two numbers are given in Figure 4.

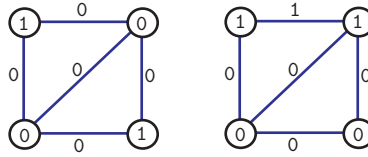


Figure 4: $PC(K_4 - e) = \{3, 5\}$.

Assume $n \geq 5$. Let u, v be the two vertices of the edge e , and let $f : V(K_n - e) \rightarrow \mathbb{Z}_2$ be an arbitrary friendly labeling of $K_n - e$. We consider the following two cases:

Case I. $n = 2\ell$. Then ℓ vertices of K_n are labeled 1. If $f(u) = 0$ or $f(v) = 0$, then K_n has K_ℓ as a subgraph all of whose vertices are labeled 1. That is, the induced edge labeling of f marks exactly $|K_\ell| = \ell(\ell - 1)/2$ edges of K_n by 1. Therefore, $e_f(1) = \ell(\ell - 1)/2$.

If $f(u) = f(v) = 1$, then K_n has $K_\ell - e$ as a subgraph all of whose vertices are labeled 1. That is, the induced edge labeling of f marks exactly $|K_\ell - e|$ edges of K_n by 1. Therefore, $e_f(1) = -1 + \ell(\ell - 1)/2$.

In this case, the only two elements of $PC(K_n)$ would be $\ell^2 \pm 1$.

Case II. Let $n = 2\ell + 1$ be odd. In this case we have two options:

(a) ℓ vertices of K_n are labeled 1 and the other $\ell + 1$ vertices are labeled 0. Then, by the argument presented in case I, $e_f(1) = \ell(\ell - 1)/2$ or $e_f(1) = -1 + \ell(\ell - 1)/2$.

(b) $\ell + 1$ vertices of K_n are labeled 1 and the other ℓ vertices are labeled 0. Then, by a similar argument that is presented in case I, $e_f(1) = \ell(\ell + 1)/2$ or $e_f(1) = -1 + \ell(\ell + 1)/2$.

Therefore, in this case, there are exactly four elements in $PC(K_n)$, which are $\ell^2 \pm 1$ and $\ell^2 + 2\ell \pm 1$. $\square$

Theorem 2.3. *The pc-set of any bipartite graph $K_{m,n}$ is*

$$\left\{ mn + 2i^2 - i(m+n) \pm \frac{i}{2} \left(1 - (-1)^i \right)^{m+n} : 0 \leq i \leq \left\lfloor \frac{m+n+1}{4} \right\rfloor \right\}.$$

Proof. Let $S = \{u_1, u_2, \dots, u_m\}$ and $T = \{v_1, v_2, \dots, v_n\}$ be the two partite sets. Due to symmetry of $K_{m,n}$, we may assume $m \leq n$. The graph $K_{m,n}$ has $m+n$ vertices and mn edges. Consider the friendly labeling $f : V(K_{m,n}) \rightarrow \mathbb{Z}_2$ of $K_{m,n}$ defined by $f(u_1) = f(u_2) = \dots = f(u_i) = 1$, $f(v_1) = f(v_2) = \dots = f(v_j) = 1$, and label the remaining vertices by 0. We consider the following two cases:

- (a) Let $m+n$ be even. Then $2(i+j) = m+n$ and $e_f(1) = ij$. Therefore, $pc(f) = mn - 2ij = mn + 2i^2 - i(m+n)$, where $i = 0, 1, 2, \dots, m$.
- (b) Let $m+n$ be odd. In this case $2(i+j) = m+n \pm 1$ and $pc(f) = mn - 2ij = mn + 2i^2 - i(m+n) \pm i$, where $i = 0, 1, 2, \dots, m$.

Finally, we observe that if we define $h(i) = mn + 2i^2 - i(m+n)$, then $h(\frac{m+n}{2} - i) = h(i)$. Therefore, to generate all the elements of $PC(K_{m,n})$, it is enough to let $i = 0, 1, \dots, \lfloor \frac{m+n+1}{4} \rfloor$. $\square$

3 PC-Sets of Trees

The complete bipartite graphs $K_{1,n}$ is also known as stars. By theorem 2.3 we know that:

Theorem 3.1. $PC(K_{1,n}) = \begin{cases} \{1, n\} & \text{if } n \text{ is odd;} \\ \{0, 2, n\} & \text{if } n \text{ is even.} \end{cases}$

Three different friendly labelings of $K_{1,6}$ that provide its pc-set are illustrated in Figure 5.

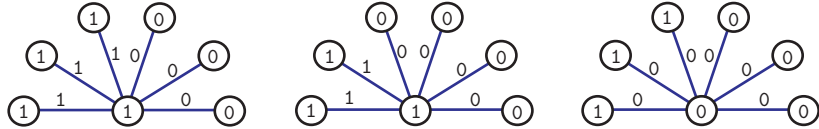


Figure 5: $PC(K_{1,6}) = \{0, 2, 6\}$.

A *double star* is a tree of diameter 3. Double stars have two central vertices u and v and are denoted by $DS(m, n)$, where $\deg u = m$ and $\deg v = n$, as illustrated in Figure 6.

Theorem 3.2. *The pc-set of double-star $DS(m, n)$ with $m \leq n$ is*

$$\left\{ m + n - 1 - 2k : 0 \leq k \leq m-1 \text{ or } \left\lfloor \frac{n-m}{2} \right\rfloor \leq k \leq \left\lfloor \frac{m+n-1}{2} \right\rfloor \right\}.$$

Proof. Consider $G = DS(m, n)$, which has $m+n$ vertices, hence $m+n-1$ edges. Note that $DS(1, 1) \cong P_2$, $DS(1, 2) \cong DS(2, 1) \cong P_3$, $DS(1, n) \cong DS(n, 1) \cong ST(n)$ and their pc-numbers are

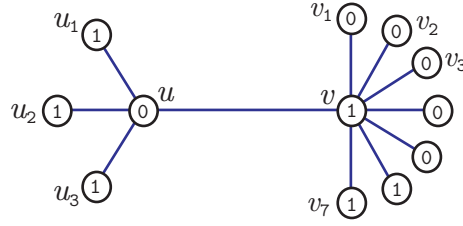


Figure 6: One of $DS(4, 8)$ cordial labelings with pc-number 7.

calculated in theorems 1.3 and 3.1. Furthermore, the results are consistent with the statement of this theorem.

Assume that $2 \leq m \leq n$ and let $f : V(G) \rightarrow \mathbb{Z}_2$ be a friendly labeling of G with $f(u_1) = f(u_2) = \dots = f(u_i) = 1$ and $f(v_1) = f(v_2) = \dots = f(v_j) = 1$. For the labels of the central vertices u and v we have the following four options:

Option 1. Let $f(u) = f(v) = 0$. Then $e_f(1) = 0$ and $pc(f) = m + n - 1$.

Option 2. Let $f(u) = f(v) = 1$. Then $e_f(1) = i + j + 1$ and $pc(f) = m + n - 1 - 2(i + j + 1)$. Now if $m + n$ is even, then $2(i + j + 2) = m + n$ and therefore $pc(f) = 1$. However, if $m + n$ be odd, then $2(i + j + 2) = m + n \pm 1$, which results in $pc(f) = 0, 2$.

Option 3. Let $f(u) = 1$, $f(v) = 0$. Then $e_f(1) = i$ and $pc(f) = m + n - 1 - 2i$, with $0 \leq i \leq m - 1$.

Option 4. Let $f(u) = 0$, $f(v) = 1$. Then $e_f(1) = j$ and $pc(f) = m + n - 1 - 2j$, with $\lfloor (n - m)/2 \rfloor \leq j \leq \lfloor (m + n - 1)/2 \rfloor$.

These four options imply that the pc-set of $DS(m, n)$ is a subset of $\{m + n - 1 - 2k : 0 \leq k \leq m - 1 \text{ or } \lfloor (n - m)/2 \rfloor \leq k \leq \lfloor (m + n - 1)/2 \rfloor\}$. On the other hand, every friendly labeling of $DS(m, n)$ is among one of the above options. Therefore, the equality holds. $\square$

Corollary 3.3. *The double-star $DS(m, n)$ is fully pc if and only if $n \leq 3m + 1$.*

Proof. We note that $DS(m, n)$ is fully pc if and only if $m \geq \lfloor (n - m)/2 \rfloor$, which implies that $n \leq 3m + 1$. Conversely, if $n \leq 3m + 1$, then $(n - m)/2 \leq m + 1/2$ and $\lfloor (n - m)/2 \rfloor \leq m$. $\square$

Examples 3.4.

- (a) The graph $DS(m, m)$ has $2m$ vertices and $2m - 1$ edges. Therefore, $PC(DS(m, m)) = \{2m - 1 - 2k : 0 \leq k \leq m - 1\} = \{1, 3, \dots, 2m - 1\}$.
- (b) The graph $DS(m, m + 1)$ has $2m$ edges. Hence, $PC(DS(m, m + 1)) = \{2m - 2k : 0 \leq k \leq m\} = \{0, 2, \dots, 2m\}$.
- (c) $PC(DS(3, 7)) = \{9 - 2k : 0 \leq k \leq 4\} = \{1, 3, 5, 7, 9\}$, and $DS(3, 7)$ is fully product cordial.

- (d) $PC(DS(3, 13)) = \{15 - 2k : 0 \leq k \leq 2 \text{ or } 5 \leq k \leq 7\} = \{1, 3, 5, 11, 13, 15\}$ and $DS(3, 13)$ is not fully product cordial.

4 PC Sets of Cycles and Cycle Related Graphs

Theorem 4.1. *For any $n \geq 3$, the graph C_n is fully product-cordial. That is,*

$$PC(C_n) = \{n - 2k : 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\}.$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of C_n . We consider two cases:

- (a) $n = 2\ell$. We show that for any $j \in \{0, 1, \dots, \ell - 1\}$ there is a friendly labeling of C_n whose pc-index is j . To achieve this, we label $\ell + 1$ consecutive vertices $v_1, v_2, \dots, v_{j+1}$ by 1 then alternate the labels 0 and 1 and label all the remaining vertices by 0. The induced edge labeling labels j edges with 1 and the rest with 0.
- (b) $n = 2\ell + 1$. With a similar argument, one can show that for any $j \in \{0, 1, \dots, \ell\}$ there is a friendly labeling of C_n whose pc-index is j .

Three different friendly labelings of C_6 that provide its pc-set are illustrated in Figure 7.

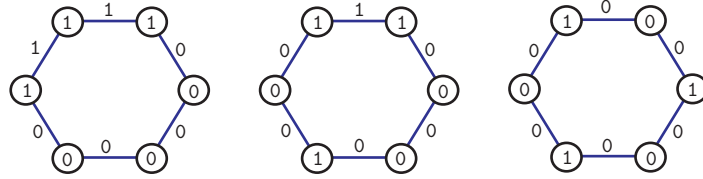


Figure 7: $PC(C_6) = \{2, 4, 6\}$.

For $n \geq 3$, the wheel W_n is the graph $C_n + K_1$.

Theorem 4.2. *For each integer $n \geq 3$,*

$$PC(W_n) = \left\{ 2n - 2j : \frac{1 - (-1)^n}{2} \leq j \leq 2\lfloor n/2 \rfloor - 1 \right\}.$$

Proof. Construct $W_n = C_n + K_1$ from an n -cycle $C_n : u_1, u_2, \dots, u_n, u_1$ by adding a new vertex v (central vertex) and joining v to every vertex of C_n . Note that W_n has $n + 1$ vertices and $2n$ edges. First, we observe that the maximum value of $e_f(1)$ occurs when the central vertex is labeled 1 and $\lfloor n/2 \rfloor$ consecutive vertices of C_n are labeled 1. The induced edge labeling of this particular friendly labeling marks $2\lfloor n/2 \rfloor - 1$ edges of W_n by 1. Therefore, for any friendly labeling $f : V(W_n) \rightarrow \mathbb{Z}_2$, we have $e_f(1) \leq 2\lfloor n/2 \rfloor - 1$. Now, based on the parity of n , we consider the following two cases:

Case 1. The cycle has even number of vertices. In this case, the statement of the Theorem becomes $PC(W_{2n}) = \{4n - 2j : 0 \leq j \leq 2n - 1\}$. We now show that for every $j = 0, 1, 2, \dots, 2n - 1$, there is a friendly labeling $f : V(W_{2n}) \rightarrow \mathbb{Z}_2$ with $e_f(1) = j$.

Subcase 1A. $0 \leq j \leq n - 1$. We define $f : V(W_{2n}) \rightarrow \mathbb{Z}_2$ by $f(v) = 0$ and

$$f(u_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq j + 1; \\ (1 - (-1)^{i-j})/2 & \text{if } j + 1 < i < 2n - j; \\ 0 & \text{if } 2n - j \leq i \leq 2n. \end{cases}$$

This coloring labels n vertices of C_{2n} by 1 and the other n vertices of C_{2n} by 0. Hence, it is a friendly labeling of W_{2n} . The induced edge labeling of this function marks exactly j edges of W_{2n} by 1. That is, $e_f(1) = j$.

Subcase 1B. $n \leq j \leq 2n - 1$. Let $g : V(W_{2n}) \rightarrow \mathbb{Z}_2$ be the inverse coloring of f defined by $g(x) = 1 - f(x) \forall x \in V(W_{2n})$. The coloring g labels $n + 1$ vertices of W_{2n} by 1 and the other n vertices by 0. Hence, g is a friendly labeling. Moreover, $pc(g) = n + pc(f)$. When $pc(f)$ changes from 0 to $n - 1$, then $pc(g)$ covers the numbers n through $2n - 1$. This completes the proof in this case.

Six different friendly labelings of W_6 that provide its pc-set are illustrated in Figure 8, the labelings in each row are the inverse labelings of the other row.

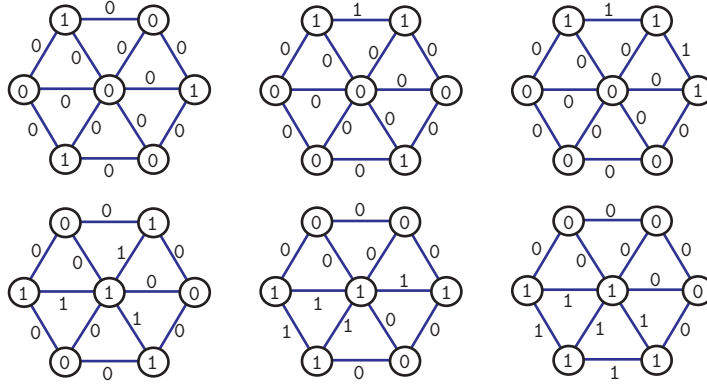


Figure 8: $PC(W_6) = \{2, 4, 6, 8, 10, 12\}$.

Case 2. The cycle has odd number of vertices. First, we observe that the minimum value of $PC(W_{2n+1})$ occurs when the central vertex v and n vertices of the cycle C_{2n+1} are labeled 0. As a result, the other $n + 1$ vertices of C_{2n+1} get label 1. Consequently, at least two adjacent vertices of C_{2n+1} get label 1 and the induced edge labeling of this cordial labeling marks at least one edge of W_{2n+1} with 1. Therefore, for any friendly labeling $f : V(W_{2n+1}) \rightarrow \mathbb{Z}_2$ we have $1 \leq e_f(1) \leq 2n - 1$. Next, note that in this case, the statement of the Theorem becomes $PC(W_{2n+1}) = \{4n + 2 - 2j : 1 \leq j \leq 2n - 1\}$. To prove this, a similar scheme that was used in case 1 can be utilized to show that for every $j = 1, 2, \dots, 2n - 1$, there is a friendly labeling $f : V(W_{2n+1}) \rightarrow \mathbb{Z}_2$ with $e_f(1) = j$. Three different friendly labelings of W_5 that provide its pc-set are illustrated in Figure 9. $\square$

Corollary 4.3. For $n \geq 3$, the wheel W_n is not fully product-cordial.

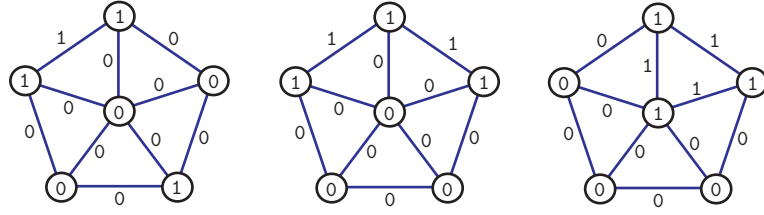


Figure 9: $PC(W_5) = \{4, 6, 8\}$.

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