

Product Cordial Sets of Trees

Ebrahim Salehi, Seth Churchman, Tahj Hill, Jim Jordan

Department of Mathematical Sciences
University of Nevada, Las Vegas
Las Vegas, NV 89154-4020
ebrahim.salehi@unlv.edu

Abstract

A binary vertex coloring (labeling) $f : V(G) \rightarrow \mathbb{Z}_2$ of a graph G is said to be friendly if the number of vertices labeled 0 is almost the same as the number of vertices labeled 1. This friendly labeling induces an edge labeling $f^* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(uv) = f(u)f(v)$ for all $uv \in E(G)$. Let $e_f(i) = |\{uv \in E(G) : f^*(uv) = i\}|$ be the number of edges of G that are labeled i . Product-cordial index of the labeling f is the number $pc(f) = |e_f(0) - e_f(1)|$. The product-cordial set of the graph G , denoted by $PC(G)$, is defined by

$$PC(G) = \{pc(f) : f \text{ is a friendly labeling of } G\}.$$

In this paper, we determine the product-cordial sets of certain classes of trees.

Key Words: friendly coloring, product-cordial index, product-cordial set, fully cordial.

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1 Introduction

In this paper all graphs $G = (V, E)$ are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [6]. Let G be a graph and $f : V(G) \rightarrow \mathbb{Z}_2$ be a binary vertex coloring (labeling) of G . For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$. The coloring f is said to be *friendly* if $|v_f(1) - v_f(0)| \leq 1$. That is, the number of vertices colored 0 is almost the same as the number of vertices colored 1.

Any friendly coloring $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(xy) = f(x)f(y) \forall xy \in E(G)$. For $i \in \mathbb{Z}_2$, let $e_f(i) = |f^{*-1}(i)|$ be the number of edges of G that are labeled i . The number $pc(f) = |e_f(1) - e_f(0)|$ is called the *product-cordial index (or pc-index)* of f . The *product-cordial set (or pc-set)* of the graph G , denoted by $PC(G)$, is defined by

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$$PC(G) = \{pc(f) : f \text{ is a friendly vertex coloring of } G\}.$$

To illustrate the above concepts, consider the graph G of Figure 1, which has 9 vertices. The condition $|v_f(1) - v_f(0)| \leq 1$ implies that four vertices be labeled 0 and the other five 1 or vice versa.

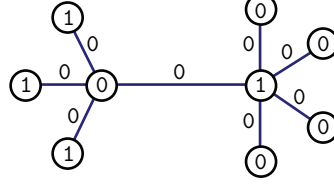


Figure 1: An example of product-cordial labeling of G .

Figure 1 also shows the associated edge labeling of G , where all edges have label 0. Therefore, the product-cordial index (or pc-index) of this labeling is $e(0) - e(1) = 8 - 0 = 8$. It is easy to see that $PC(G) = \{0, 2, 4, 6, 8\}$. The friendly colorings of G that provide the other four pc-indices are presented in Figure 2.

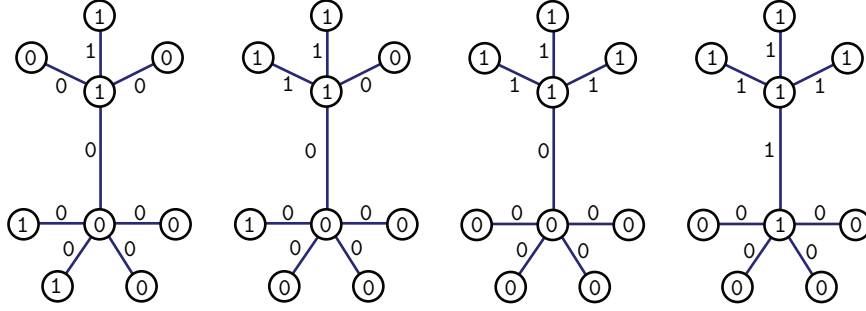


Figure 2: Four friendly labelings of G with pc-indices 6, 4, 2 and 0.

In 1978, I. Cahit [2, 3, 4] introduced the concept of cordial labeling as weakened version of the less tractable graceful and harmonious labeling. A graph G is said to be cordial if it admits a friendly labeling with index 0 or 1. M. Hovay [9], later generalized the concept of cordial graphs and introduced A -cordial labelings, where A is an abelian group. A graph G is said to be A -cordial if it admits a labeling $f : V(G) \rightarrow A$ such that for every $i, j \in A$,

$$|v_f(i) - v_f(j)| \leq 1 \text{ and } |e_f(i) - e_f(j)| \leq 1.$$

Cordial graphs have been studied extensively. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [1, 5, 8, 10, 11, 14, 20].

Product cordial labeling of a graph was introduced by Sundaram, Ponraj and Somasundaram [23]. They call a graph G product-cordial if it admits a friendly labeling whose product-cordial index is at most 1. Then Sundaram, Ponraj and Somasundaram [23, 24, 25] investigated whether certain graphs such as trees, cycles, complete graphs, wheels, etc. are product-cordial. Later E. Salehi [15] introduced the concept of product-cordial set (or pc-set) of a graph and determined the pc-sets of certain classes of graphs such as: complete graphs, complete bipartite graphs, stars and double stars, cycles, and wheels. In this paper we determine the product cordial sets of certain classes of trees.

In what follows, whenever there is no ambiguity, we suppress the index f and denote $e_f(i)$ by simply $e(i)$. For a graph $G = (p, q)$ of size q , and a friendly labeling $f : V(G) \rightarrow \mathbb{Z}_2$ of G , we have

$$pc(f) = |e_f(0) - e_f(1)| = |q - 2e_f(1)| = |q - 2e_f(0)|. \quad (1.1)$$

Therefore, to find the pc-index of f it is enough to find $e_f(1)$ (or $e_f(0)$). Moreover, to determine the pc-set of G it is enough to compute $e_f(1)$ for different friendly colorings of G . Another immediate consequence of (1.1) is the following useful fact:

Observation 1.1. *For a graph G of size q , $PC(G) \subseteq \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}$.*

Definition 1.2. *A graph G of size q is said to be fully product-cordial (fully pc) if*

$$PC(G) = \{q - 2k : 0 \leq k \leq \lfloor q/2 \rfloor\}.$$

For example, the graph G of Figure 1 is not fully pc. However, P_n , the path of order n , is fully pc. In case of P_n , it is easy to see that there are friendly colorings of P_n such that $e(1) = 0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor$, which proves the following theorem:

Theorem 1.3. *For any $n \geq 2$, the graph P_n is fully product-cordial. That is, $PC(P_n) = \{n - 1 - 2k : 0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor\}$.*

The different friendly labelings of P_7 that provide its pc-set are illustrated in Figure 3.

Definition 1.4. A *matching* in a graph is a set of edges with no shared endpoints. A matching M in a graph G is said to be a *perfect matching* if every vertex of G is incident with an edge in M .

Theorem 1.5. [16] *Any tree T of order p with a perfect matching is fully product-cordial. That is,*

$$PC(T) = \{1, 3, 5, \dots, p - 1\}.$$

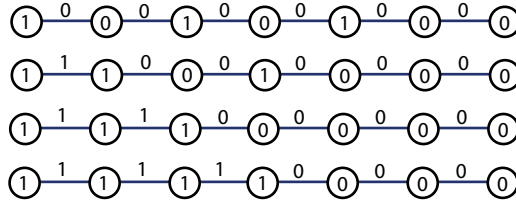


Figure 3: $PC(P_7) = \{0, 2, 4, 6\}$.

In general, for a friendly coloring $f : V(G) \rightarrow \mathbb{Z}_2$ of a graph G , it is not necessarily true that $e_f(0) \geq e_f(1)$. For example, let $n > 3$ and consider the coronation [7] of the complete graph K_n with K_1 , as indicated in Figure 4.

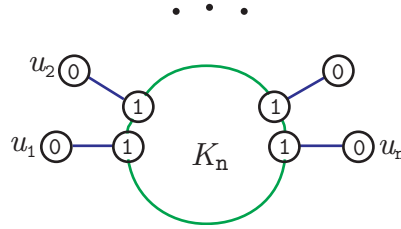


Figure 4: A friendly coloring with $e(1) > e(0)$.

If we color all vertices of K_n by 1 and the end-vertices by 0, then $e(1) = n(n-1)/2$ while $e(0) = n$. However, for certain graphs one can prove that the number of edges labeled 0 is not less than the number of edges labeled 1. Trees are among such graphs as we will see in the following theorem:

Theorem 1.6. *For any tree T and any friendly coloring of T , $e(0) \geq e(1)$.*

Proof. The statement is true for trees of order $n = 1, 2, 3$. Let T be a tree of order $n \geq 4$ and observe that at least $e(1) + 1$ vertices of T are labeled with 1. Since the coloring is friendly, at least $e(1)$ vertices of T are labeled with 0. This implies that $n \geq 2e(1) + 1$ or $|E| \geq 2e(1)$. Therefore, $2e(1) \leq |E| = e(1) + e(0)$, or $e(1) \leq e(0)$. $\square$

One immediate consequence of Theorem 1.6 and equation (1.1) is that for any coloring $f : T \rightarrow \mathbb{Z}_2$ of a tree T , $pc(f) = q - 2e_f(1)$. As a result, to show that a tree T is fully product cordial, one needs to present different friendly colorings of T such that $e(1) = 0, 1, \dots, \lfloor \frac{q}{2} \rfloor$, where q is the number of edges of T . This observation leads us to the following useful fact:

Theorem 1.7. Consider the tree $T = (p, q)$ and for any $j = 0, 1, \dots, \lfloor \frac{q}{2} \rfloor$ let S_j be a subtree of T with $j + 1$ vertices. Label all vertices of S_j with 1. If we can extend this labeling to a friendly labeling of T such that $e(1) = j$, then T is fully product cordial.

This theorem is applied to P_7 , as illustrated in Figure 3.

Observation 1.8. If T is a tree of odd (even) order, then there is a friendly coloring of T with pc index 0 (1).

Proof. Let T be tree of order n and choose a subtree S of T of order $\lceil n/2 \rceil$. If we label all the vertices of S by 1 and the remaining vertices of T by 0, then this is a friendly coloring of T with $e(0) - e(1) = \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor$. $\square$

Next, we show that any full binary tree is fully cordial.

Definition 1.9. Full binary tree of depth d , denoted by FB_d , is defined inductively as follows: FB_1 is the path P_3 with the middle vertex being its root and for $n \geq 2$, FB_n is a binary tree of root r_n whose left and right children are FB_{n-1} .

Note that in a full binary tree of depth d , there are $2^{d+1} - 1$ vertices, hence $2^{d+1} - 2$ edges. Such a tree contains 2^d leaves (end-vertices) all of them in the d^{th} row and there are $2^d - 1$ vertices with degrees bigger than 1.

Theorem 1.10. Any full binary tree is fully product cordial. Moreover, every index can be obtained by a friendly coloring that labels half of the leaves 1 and the other half 0.

Proof. We proceed by induction on d , the depth of the full binary tree. For convenient, for any non-root vertex u , its parent vertex will be denoted by p_u . Clearly the statement is true for $d = 1, 2$ as illustrated in Figure 5.

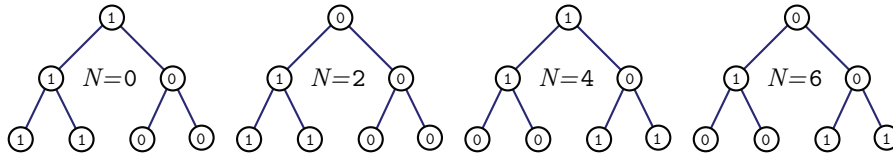


Figure 5: Four friendly colorings of FB_2 with four different indices.

Suppose the statement is true for FB_{d-1} . That is, $PC(FB_{d-1}) = \{0, 2, 4, \dots, 2^d - 2\}$ and every member of this set can be obtained by a friendly coloring that labels 2^{d-1} leaves 1 and the other

2^{d-1} leaves 0. We wish to show that $PC(FB_d) = \{0, 2, 4, \dots, 2^d - 2, 2^d, \dots, 2^{d+1} - 2\}$. Let j be any non-negative even integer less than $2^{d+1} - 1$. We consider two cases:

- A.** $0 \leq j \leq 2^d - 2$. In this case, let f be the friendly coloring of FB_{d-1} that produces index j and extend it to $\varphi : V(FB_d) \rightarrow \mathbb{Z}_2$ by defining

$$\varphi(u) = \begin{cases} f(u) & \text{if } \deg u > 1; \\ f(u_p) & \text{if } \deg u = 1. \end{cases}$$

Then φ is a friendly coloring of FB_d with $e_\varphi(1) = e_f(1) + 2^{d-1}$, $e_\varphi(0) = e_f(0) + 2^{d-1}$. Therefore, $pc(\varphi) = e_\varphi(0) - e_\varphi(1) = pc(f) = j$.

- B.** $2^d \leq j \leq 2^{d+1} - 2$. In this case, let g be the friendly coloring of FB_{d-1} that produces index $j - 2^d$ and extend it to $\psi : V(FB_d) \rightarrow \mathbb{Z}_2$ by defining

$$\psi(u) = \begin{cases} f(u) & \text{if } \deg u > 1; \\ 1 - f(u_p) & \text{if } \deg u = 1. \end{cases}$$

Then ψ is a friendly coloring of FB_d with $e_\psi(1) = e_f(1)$, $e_\psi(0) = e_f(0) + 2^d$. Therefore, $pc(\psi) = e_\psi(0) - e_\psi(1) = pc(f) + 2^d = j$.

Moreover, not only φ, ψ are friendly colorings of FB_d , but they also label 2^{d-1} leaves 1 and the other 2^{d-1} leaves 0. □

2 Near Perfect Matching Trees

In [16], Salehi-Mukhin showed that any tree with perfect matching is fully product cordial. However, there are many other fully pc trees that do not have perfect matchings. Paths of odd orders P_{2n+1} are the most obvious examples. In this section we introduce another class of fully pc trees. Namely, near perfect matching trees.

Definition 2.1. A matching of a graph G is called *near perfect matching* if it covers all the vertices of G but one. G is called a *near perfect matching graph* if any maximum matching of G is near perfect matching.

Observation 2.2. A tree T with near perfect matching M contains at least a P_3 pendent $u_1 \sim u_2 \sim u_3$ such that $\deg u_1 = 1$, $\deg u_2 = 2$ and $u_1 u_2 \in M$.

Proof. Let $P : u_1 \sim u_2 \sim u_3 \sim \cdots u_{k-2} \sim u_{k-1} \sim u_k$ be the longest path in T . Clearly, $\deg u_1 = \deg u_k = 1$. Also, $\deg u_2 = 2$ or $\deg u_{k-1} = 2$. Otherwise, any maximum matching of T would miss at least two vertices. If $\deg u_2 = \deg u_{k-1} = 2$, then $u_1 u_2 \in M$ or $u_k u_{k-1} \in M$. Suppose (wlog) $\deg u_2 = 2$ and $\deg u_{k-1} > 2$. Then $u_1 u_2 \in M$. Otherwise, any maximum matching of T would miss at least two vertices. $\square$

Theorem 2.3. *Any tree T with near perfect matching is fully product-cordial.*

Proof. Note that T is a tree of odd order, $|T| = 2n + 1$. We proceed by induction on n . Clearly, the statement of theorem is true for $n = 1$. Suppose the statement is true for any tree of order $2n + 1$ and let T be a tree of order $2n + 3$ with near-perfect matching M . By 2.2, T contains vertices $u \sim v \sim w$ such that $\deg w = 1$, $\deg v = 2$ and the edge vw is in M . Now consider the tree $S = T - \{v, w\}$ which has order $2n + 1$ and has near perfect matching $M' = M - \{vw\}$. Therefore, by the induction hypothesis

$$PC(S) = \{0, 2, 4, \dots, 2n\}.$$

We need to show that $PC(T) = \{0, 2, \dots, 2n, 2n + 2\}$. Consider a friendly coloring $f : V(S) \rightarrow \mathbb{Z}_2$ of S and extend it to $\varphi : V(T) \rightarrow \mathbb{Z}_2$ by defining

$$\varphi(x) = \begin{cases} f(x) & \text{if } x \neq v, w; \\ f(u) & \text{if } x = w; \\ 1 - f(u) & \text{if } x = v. \end{cases}$$

Then φ is a friendly coloring of T with $e_\varphi(1) = e_f(1)$, $e_\varphi(0) = e_f(0) + 2$. Therefore, $pc(\varphi) = pc(f) + 2$. This implies that

$$2 + PC(S) = \{2, 4, \dots, 2n + 2\} \subseteq PC(T).$$

It only remains to show that $0 \in PC(T)$, which follows from 1.8. $\square$

Definition 2.4. *Fibonacci Trees*, denoted by FT_n , are defined inductively as follows: FT_1 is the trivial tree with one vertex, FT_2 is the path P_2 , and for $n \geq 3$, $FT_n = (V_n, E_n)$ is the binary tree of root r_n , whose left and right children are FT_{n-1} and FT_{n-2} , respectively.

There is a relationship between the order of FT_n and Fibonacci numbers. In fact, the number of vertices of FT_n is $A_{n+2} - 1$, where A_n is the n^{th} Fibonacci number.

Theorem 2.5. *For $n \geq 1$, every Fibonacci tree FT_n is fully product cordial.*

Proof. Note that every Fibonacci tree has either a perfect matching or is a near perfect matching tree. In fact, if $n \equiv 1 \pmod{3}$, then FT_n is a near perfect matching tree; Otherwise, it has a

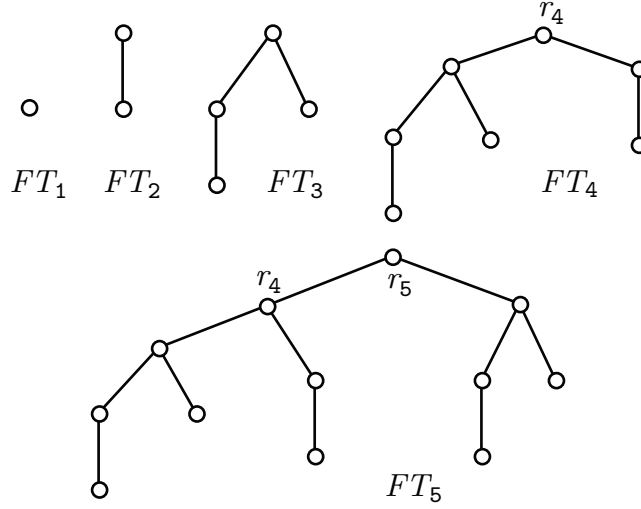


Figure 6: Graphs of the first five Fibonacci trees.

perfect matching. We prove this statement by induction on n . Clearly the statement is true for $n = 1, 2, 3$. Now suppose the statement is true for all positive integers less than n ($3 < n$) and let FT_n be the Fibonacci tree of order n . We consider the following cases:

- (A) $n \equiv 1 \pmod{3}$. In this case, by the induction hypothesis, both the left and right children have perfect matchings. Let M_1 and M_2 be perfect matchings of FT_{n-1} and FT_{n-2} , respectively. Then $M_1 \cup M_2$ is a maximum matching of FT_n that covers all the vertices but its root. Therefore, FT_n is a near perfect matching tree.
- (B) $n \equiv 2 \pmod{3}$. In this case, by the induction hypothesis, the left child FT_{n-1} is near perfect matching while the right child FT_{n-2} has a perfect matching. Let M_1 be a maximum matching of FT_{n-1} (we may assume that M_1 leaves the root r_{n-1} out) and M_2 be a perfect matching of FT_{n-2} . Then $M_1 \cup M_2 \cup \{r_n r_{n-1}\}$ will form a perfect matching of FT_n .
- (C) $n \equiv 0 \pmod{3}$. The argument is similar to the previous case.

3 Lucas Trees

Definition 3.1. *Lucas Trees, denoted by LT_n , are defined inductively as follows: LT_1 is the trivial tree with one vertex, LT_2 is the path P_3 with the middle vertex being its root, and for $n \geq 3$, $LT_n = (V_n, E_n)$ is the binary tree of root r_n , whose left and right children are LT_{n-1} and LT_{n-2} , respectively.*

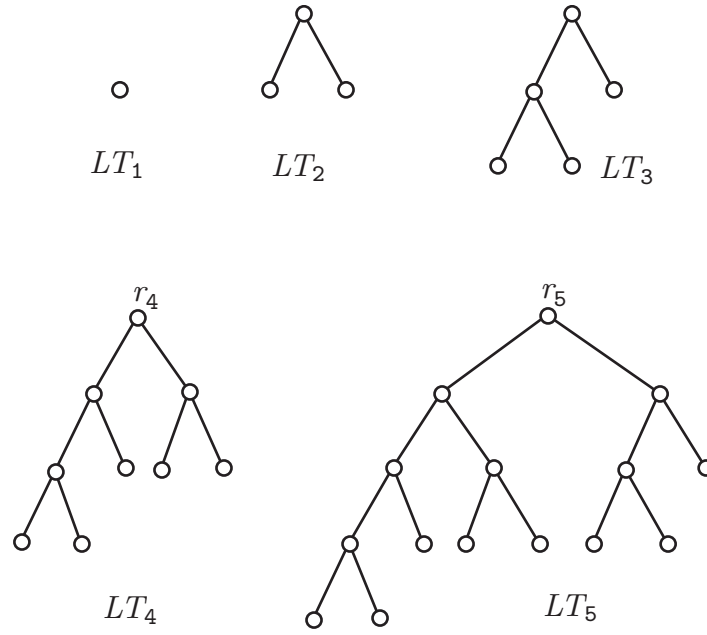


Figure 7: The first five Lucas trees.

Theorem 3.2. For $n \geq 1$, every Lucas tree LT_n is fully product cordial.

Proof. This is an easy consequence of theorem 1.7. □

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